

# Effects of Spectral Amplitude and Phase Errors on Interpretability of Images

W. H. Hsiao<sup>†</sup>, E. Ip<sup>§</sup> and R. P. Millane<sup>†</sup>

<sup>†</sup>Department of Electrical and Computer Engineering  
University of Canterbury  
Private Bag 4800, Christchurch, New Zealand  
rick@elec.canterbury.ac.nz

<sup>§</sup> Department of Electrical Engineering  
Stanford University  
Stanford, California, USA

## Abstract

The relative effects of spectral amplitude and phase errors on reconstructed images is studied in terms of the expected mean-square-error in the image. An appropriate mean-square-error appears to be that between reconstructed and original images that are scaled to have the same energy. Such an error metric appears to reflect the overall perceived quality of the images. Approximate relationships between spectral amplitude and phase errors that give rise to the same image mean-square-error are derived. Simulations are used to illustrate these relationships. The relationship to phase dominance is discussed.

**Keywords:** spectrum, Fourier transform, phase, amplitude, image reconstruction, phase dominance

## 1 Introduction

In many imaging, remote sensing and image processing problems, the Fourier transform, or spectrum, of an image, rather than the image itself, is measured. Since the spectrum is complex, both the amplitude and the phase are needed in order to calculate the image by inverse Fourier transformation. However, in a number of applications one measures the amplitude, but not the phase, of the transform of an image. This can arise if a wavefield is measured after propagation through a random medium (that introduces large phase errors) or if the wavelength of the radiation is too small for coherent detection. "Phase retrieval" refers to the process of reconstructing an image, or equivalently the Fourier phase, from measurements of the Fourier amplitude [1, 2, 3]. Phase retrieval is very important in a number of technical fields such as astronomy, medical imaging, and biology [3, 4]. Phase retrieval algorithms seek to reconstruct an image from measurement of its spectral amplitude by incorporating *a priori* information or constraints on the allowable images. The characteristics of phase problems and development of improved algorithms is an active area of research.

A characteristic of Fourier imaging that is related to phase retrieval is the phenomenon of "phase dominance" [5, 6]. Phase dominance refers to the general observation that loss of the spectral phase information tends to lead to a less recognisable image than does loss of the spectral amplitude

information. This implies that the phase contains more information than the magnitude. This characteristic of image spectra has been known for some time in the fields of crystallography, image processing, visual perception and holography, as well as in signal coding and speech [5, 7, 8]. It is also of relevance in coding, compression and phase-only holograms (kinofoms) [9]. Although this is a well known characteristic, most studies of phase dominance have so far been based only on a qualitative and subjective analysis. Phase dominance is consistent with the observation that image reconstruction from the Fourier phase is easier than reconstruction from the Fourier amplitude.

An example illustrating phase dominance is shown in Fig. 1. Two original images are shown in a and b. Image c is calculated from the spectral amplitude of image b and the spectral phase of the image a, and image d is calculated from the spectral amplitude of image a and the spectral phase of image b. Inspection of images c and d shows that they both strongly resemble the images from which their spectral phase is taken.

In this paper we examine the effects that errors in the spectral amplitude and phase have on the interpretability of images, and appropriate error metrics. The results are of relevance to phase retrieval, visual perception and coding.

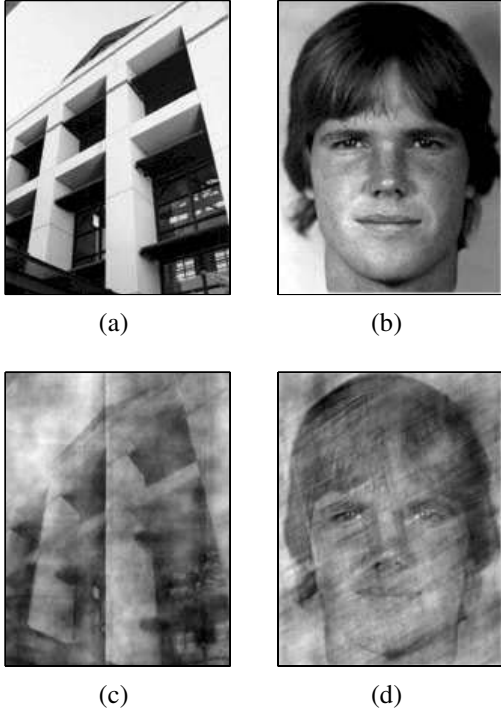


Figure 1: Illustration of phase dominance. a and b: original images. c and d: composite images calculated using the spectral phase of a and b, respectively.

## 2 Theory – Small Amplitude Errors

Consider an image  $f(x,y)$ , where  $(x,y)$  is position in image space. The Fourier transform  $F(u,v)$  of the image is given by

$$F(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) \exp(i2\pi(ux + vy)) dx dy, \quad (1)$$

where  $(u,v)$  is position in Fourier space. The Fourier transform is decomposed into the amplitude  $|F(u,v)|$  and phase  $\phi(u,v)$ , where

$$F(u,v) = |F(u,v)| \exp(i\phi(u,v)). \quad (2)$$

In order to study the importance of the spectral amplitude and phase, we consider an image denoted  $\hat{f}(x,y)$  that is reconstructed after errors have been added to the Fourier amplitude and/or phase. The effect on the image is assessed by calculating the relative mean-square-error (mse),  $e^2$ , between the reconstructed and original images, i.e.

$$e^2 = \frac{\iint [\hat{f}(x,y) - f(x,y)]^2 dx dy}{\iint [f(x,y)]^2 dx dy}. \quad (3)$$

The Fourier transform of  $\hat{f}(x,y)$  is denoted by  $\hat{F}(u,v)$  and the phase of  $\hat{F}(u,v)$  is denoted by  $\hat{\phi}(u,v)$ . Hence  $\hat{F}(u,v)$  represents  $F(u,v)$  after the addition of spectral amplitude or phase errors. The dependence of the

transform quantities on  $u$  and  $v$  is suppressed in the following where no confusion arises. The error in the transform is denoted by  $\Delta F = \hat{F} - F$ , and the errors  $\Delta|F|$  and  $\Delta\phi$  in the amplitude and phase, respectively, are given by

$$\Delta|F| = |\hat{F}| - |F| \quad (4)$$

$$\Delta\phi = \hat{\phi} - \phi. \quad (5)$$

We define the normalised variance of the amplitude errors by

$$\sigma_a^2 = \frac{\langle \Delta|F|^2 \rangle}{\langle |F|^2 \rangle}. \quad (6)$$

Assuming that the errors  $\Delta|F|$  and  $\Delta\phi$  are independent and zero-mean, we have shown that the expected mse  $\langle e^2 \rangle$  is given by [10]

$$\langle e^2 \rangle = 2\langle 1 - \cos(\Delta\phi) \rangle + \sigma_a^2, \quad (7)$$

where  $\langle \cdot \rangle$  denotes the ensemble average. Note that the mse is the sum of two terms, one which depends only on the phase errors and the other which depends only on the amplitude errors.

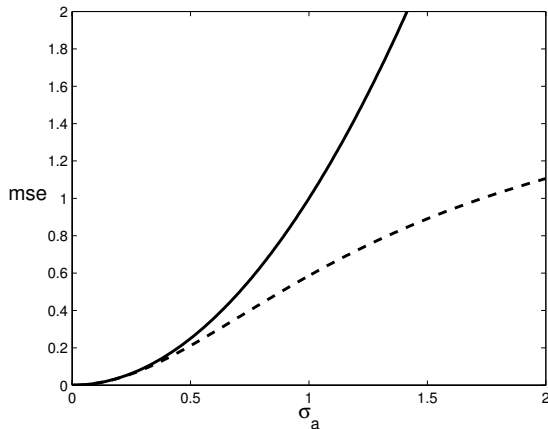
For phase errors  $\Delta\phi$  uniformly distributed between  $-A$  and  $A$  radians, the standard deviation, denoted  $\sigma_\phi$ , is  $\sigma_\phi = A/\sqrt{3}$ , and Eq. 7 can be evaluated giving [10]

$$\langle e^2 \rangle = 2 - \frac{2 \sin(\sqrt{3}\sigma_\phi)}{\sqrt{3}\sigma_\phi} + \sigma_a^2. \quad (8)$$

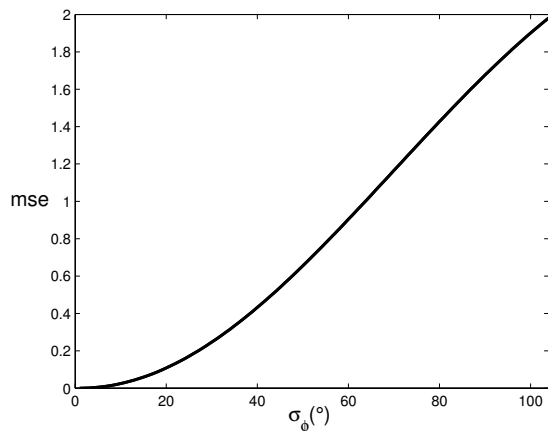
The uniform distribution is valid only for  $A < \pi$ , or  $\sigma_\phi < \pi/\sqrt{3} \approx 104^\circ$ , so we restrict  $\sigma_\phi$  to this range. Normally distributed errors were considered in [10] but are not discussed here. Plots of the mse versus the standard deviation of the amplitude and phase errors are shown in Fig. 2. These results can be used to quantify the relative effects of amplitude and phase errors by deriving a relationship between  $\sigma_a$  and  $\sigma_\phi$  such that the corresponding values give the same mse. This relationship is shown in Fig. 3.

## 3 Results – Small Amplitude Errors

Uniformly distributed amplitude and phase errors were added to the transforms of a number of images and the resulting images reconstructed. The pixel values were first shifted so that the average value over the image is zero, since the addition of phase errors to the otherwise very large value of the Fourier transform at zero spatial frequency (generally the case for images) causes unreasonably large and erratic errors in the images. The images were then scaled to the range 0 (black) to 255 (white) for display. The mses of the reconstructed images were calculated and averaged over an ensemble of noise signals. The average mses fitted the curves in Fig. 2 essentially exactly.



(a)



(b)

Figure 2: (a) The mses  $\langle e^2 \rangle$  (—) and  $\langle e_s^2 \rangle$  (---) versus  $\sigma_a$ . (b)  $\langle e^2 \rangle$  and  $\langle e_s^2 \rangle$  versus  $\sigma_\phi$ .

Examples of reconstructed images are shown in Fig. 4. The original image is shown in the top row. Each row in the figure shows reconstructed images with a constant mse, the mse increasing down the page with the values given in the caption. The images in the left column are constructed with spectral amplitude errors only, and those in the right column with phase errors only. The images in the center column are constructed using both amplitude and phase errors such that each contributes one half of the mse. Inspection of this figure shows a number of interesting features. For small values of the mse, although some distortion is present the images are quite recognisable. The effects of the amplitude and phase errors are similar although amplitude errors tend to decrease the contrast more than do the phase errors, and phase errors tend to introduce some structure not present in the original image. These effects become more pronounced as the mse increases. The bottom row of the figure ( $e^2 = 1.5$ ) corresponds to  $\sigma_a = 1.2$  for the left image and  $\sigma_\phi = 82^\circ$  for the right image. These values correspond to almost random amplitudes and phases, respectively. Although the image on the

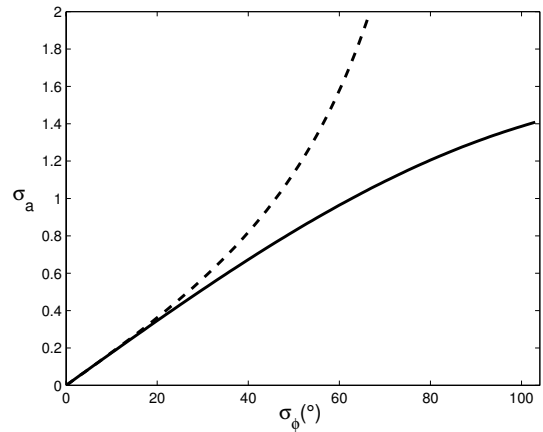


Figure 3: Relationship between amplitude error  $\sigma_a$  and phase error  $\sigma_\phi$ , for uniformly distributed errors, such that the mses  $\langle e^2 \rangle$  (—) and  $\langle e_s^2 \rangle$  (---) are equal.

left is better than that on the right, this is largely due to a scaling effect as is described in the next section. The image on the left is quite recognisable but the image on the right is not. A mixture of large amplitude and phase errors leads to a partially recognisable image (centre image in bottom row).

#### 4 Theory – Large Amplitude Errors

There are two effects that are not considered in the above analysis, that become important when the amplitude errors are large.

The first effect is that since arbitrarily small values of the amplitude  $|F|$  may occur, for any finite  $\sigma_a$  it is possible that  $\Delta|F| < -|F|$  for some samples of the spectrum. In such a case  $|\hat{F}| < 0$ , which presents a problem since an amplitude is a positive quantity. The effect in the above analysis is that the sign of any negative  $|\hat{F}|$  is changed and a phase error of  $\pi$  is introduced. In practice however, the measured amplitude would usually saturate at zero and no phase error would be introduced. Therefore, for large amplitude errors, the above analysis introduces some erroneous amplitude and phase errors. The effect increases as  $\sigma_a$  increases since the probability of negative amplitudes then increases.

The second effect can be seen by noting that  $\langle |\hat{F}|^2 \rangle = \langle |F|^2 \rangle (1 + \sigma_a^2)$ . The quantity  $\langle |F|^2 \rangle$  is a measure of the energy in the image. Hence, for large values of  $\sigma_a$  the energy in the reconstructed image is substantially larger than the energy in the original image. This tends to make the overall amplitude of the reconstructed image larger than that of the original image. The mse  $\langle e^2 \rangle$  is therefore due in part to the overall difference in amplitude between the two images. In comparing reconstructed and original images (both in terms of visual perception and in quantitative technical applications) the overall amplitude of the whole image tends

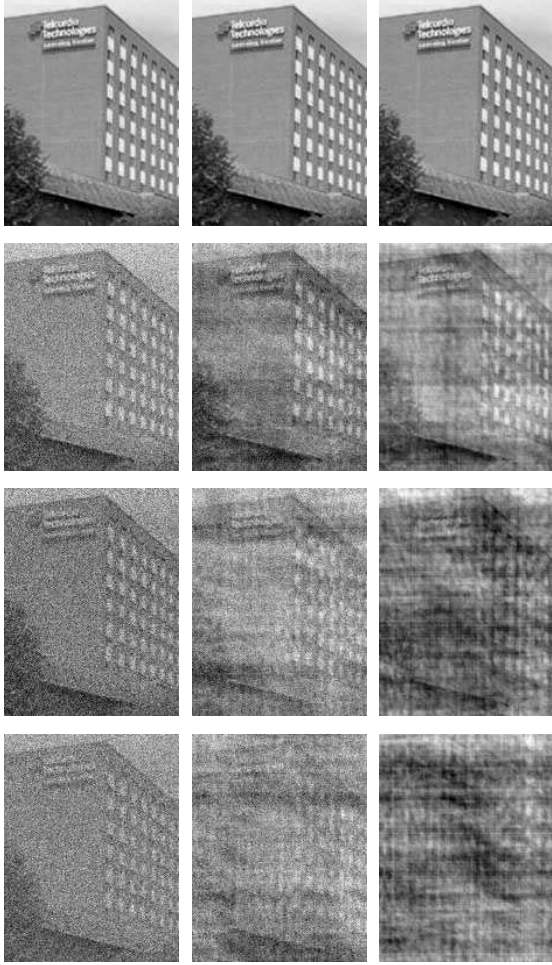


Figure 4: Images reconstructed with a variety of uniformly distributed amplitude and phase errors as described in the text. The original image is shown in the top row. The mse  $\langle e^2 \rangle$  increases down the page with values 0, 0.5, 1, 1.5. The different columns are described in the text.

to be unimportant. The effect is that the relative mse calculated by Eq. 3 overestimates the relevant error as far as image interpretability is concerned.

The second effect is more easily dealt with and so is discussed first. A more useful measure of the image error is that calculated after the image(s) have been scaled to have the same energy. We therefore define the mse  $\langle e_s^2 \rangle$  as

$$e_s^2 = \frac{\iint [\alpha \hat{f}(x,y) - f(x,y)]^2 dx dy}{\iint [f(x,y)]^2 dx dy}, \quad (9)$$

where the subscript  $s$  refers to the error after scaling the reconstructed image and

$$\alpha^2 = \frac{1}{1 + \sigma_a^2}. \quad (10)$$

The error metric  $\langle e_s^2 \rangle$  is expected to better reflect the interpretability of images than does  $\langle e^2 \rangle$ . Performing

the same analysis as in Section 2 allows  $\langle e_s^2 \rangle$  to be calculated and put in the form

$$\langle e_s^2 \rangle = \frac{2}{\sqrt{1 + \sigma_a^2}} \langle 1 - \cos(\Delta\phi) \rangle + 2 \left( 1 - \frac{1}{\sqrt{1 + \sigma_a^2}} \right) \quad (11)$$

for comparison with Eq. 7. Equation 11 is not equal to the sum of two terms which depend, respectively, on only the amplitude and phase errors, as is the case in Eq. 7. However, the first term in Eq. 11 is zero for no phase errors ( $\Delta\phi = 0$ ) and the second term is zero for no amplitude errors ( $\sigma_a = 0$ ). The first term depends on both  $\Delta\phi$  and  $\sigma_a$ , whereas the second term depends only on  $\sigma_a$ . Note that  $\langle e_s^2 \rangle$  reduces to  $\langle e^2 \rangle$  if  $\sigma_a = 0$ .

Referring to Eq. 7, the maximum value of  $\langle e^2 \rangle$  is 4 for phase errors only, but is unbounded for large amplitude errors. On the other hand, Eq. 11 gives the same upper bound for phase errors, but the mse is bounded at 2 for large amplitude errors, reflecting the normalisation of the reconstructed image. The mse is bounded by 4 if both amplitude and phase errors are present, as can be seen by writing Eq. 11 in the form

$$\langle e_s^2 \rangle = 2 \left( 1 - \frac{\langle \cos(\Delta\phi) \rangle}{\sqrt{1 + \sigma_a^2}} \right). \quad (12)$$

Evaluating 12 for uniformly distributed phase errors gives

$$\langle e_s^2 \rangle = 2 - \frac{2 \sin(\sqrt{3}\sigma_\phi)}{\sigma_\phi \sqrt{3(1 + \sigma_a^2)}}. \quad (13)$$

A plot of  $\langle e_s^2 \rangle$  versus  $\sigma_a$  for  $\sigma_\phi = 0$  is shown in Fig. 2(a). Note that the curves are similar for small  $\sigma_a$ , but that  $\langle e_s^2 \rangle$  is considerably smaller than  $\langle e^2 \rangle$  for large  $\sigma_a$ . A plot of  $\langle e_s^2 \rangle$  versus  $\sigma_\phi$  for  $\sigma_a = 0$  is identical to that for  $\langle e^2 \rangle$  (Fig. 2(b)). As described in Section 2, these results can be used to derive a relationship between  $\sigma_a$  and  $\sigma_\phi$  such that the reconstructed image have the same  $\langle e_s^2 \rangle$ . This relationship is shown in Fig. 3. Inspection of the figure shows that for a fixed  $\sigma_\phi$ , a larger  $\sigma_a$  is required to obtain the same  $\langle e_s^2 \rangle$  than is required to obtain the same  $\langle e^2 \rangle$ .

## 5 Results – Large Amplitude Errors

Reconstructed images were generated as described in Section 3 and are shown in Fig. 5, but this time with the images in a single row having the same  $\langle e_s^2 \rangle$ , rather than the same  $\langle e^2 \rangle$  as in Fig. 4. Inspection of the figure shows that the images in a single row, particularly for the lower rows, are more similar in quality than they are in Fig. 4, although the images in the left column are slightly more recognisable than those in the right

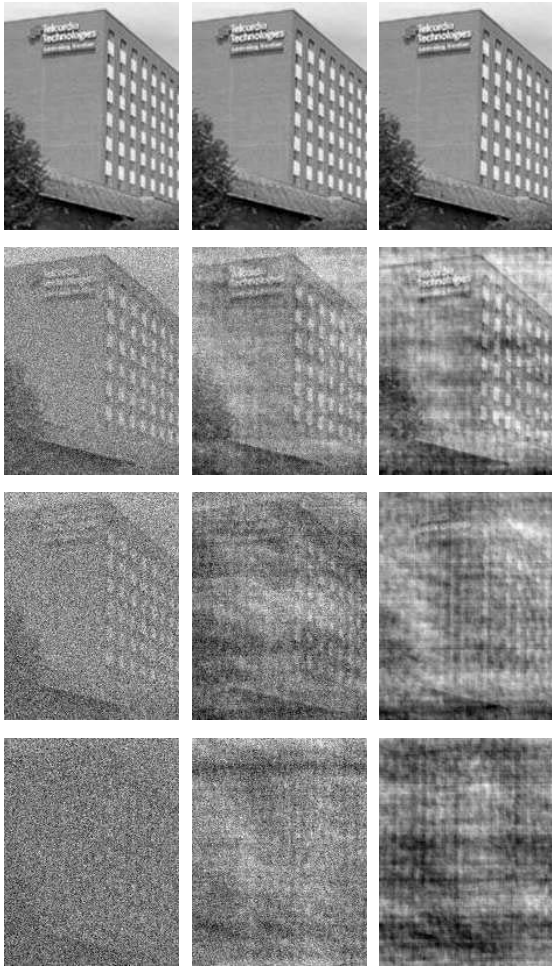


Figure 5: Images reconstructed with a variety of uniformly distributed amplitude and phase errors as described in the text. The original image is shown in the top row. The mse  $\langle e_s^2 \rangle$  increases down the page with values 0, 0.5, 1, 1.5. The different columns are described in the text.

column. The images in Fig. 5 show that when the overall amplitude of the image is taken into account, the difference between amplitude and phase errors for the same mse is less pronounced.

The analysis in Section 4 and the results shown in Fig. 5 still suffer from the problem that large amplitude errors will introduce some erroneous amplitude and phase errors into the spectrum of the reconstructed image as described in the second paragraph of Section 4. The effect of this is that images in the lower left region of Fig. 5 will generally contain smaller amplitude errors and larger phase errors, overall, than is specified by the analysis and the simulations. As described before, a more realistic model is to set any negative amplitude to zero. Derivation of an analytical expression for the mse for this model is difficult, and we investigate this case by simulation. Uniformly distributed amplitude

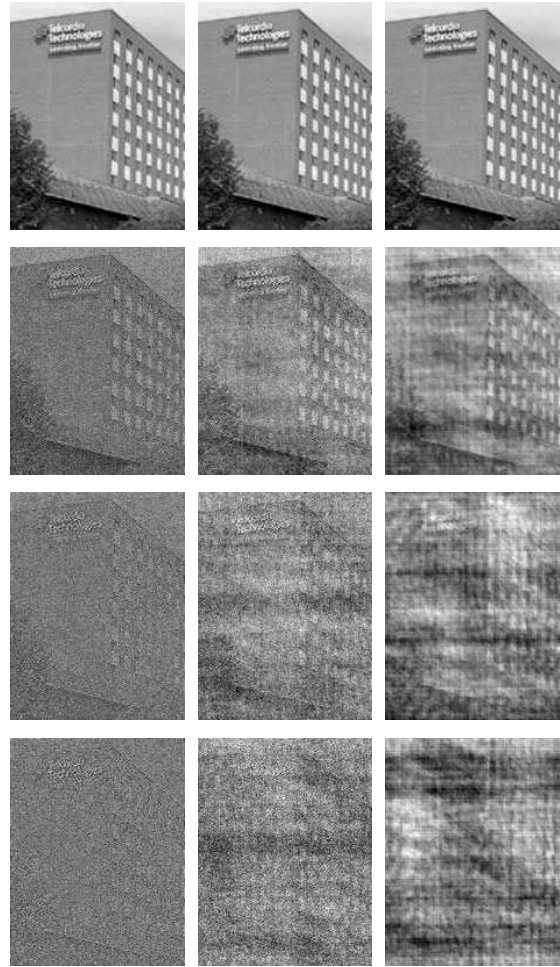


Figure 6: Images reconstructed with a variety of uniformly distributed amplitude and phase errors as described in the text. The original image is shown in the top row. The mse  $\langle e_z^2 \rangle$  increases down the page with values 0, 0.5, 1, 1.5. The different columns are described in the text.

and phase errors were added to the spectrum of an image and any negative amplitudes were set to zero. The energy of the reconstructed image was calculated and the image scaled to the energy of the original image. The mse between the reconstructed and original image, denoted by  $\langle e_z^2 \rangle$  where the subscript  $z$  denotes zeroing of the negative amplitudes, was calculated. The values of  $\sigma_a$  and/or  $\sigma_\phi$  were adjusted to give the desired values of  $\langle e_z^2 \rangle$  (as listed in the caption to Fig. 6) and the resulting images are displayed in the rows in Fig. 6. Inspection of the figure shows that the images in a single row, particularly for the lower rows, are of even more similar quality than is the case in Fig. 5. In particular, all images in the bottom row are essentially equally uninterpretable. Since this is the most realistic model of the effects of spectral amplitude and phase errors, we conclude that for a given  $\langle e_z^2 \rangle$ , amplitude and phase errors have similar effects.

## 6 Conclusions

The relative importance of spectral amplitude and phase errors on image reconstruction from the Fourier transform is of relevance to a number of technical areas including remote sensing, compression and visual perception. The relative effects of amplitude and phase errors can be evaluated by considering errors that give identical mean-square-errors in a reconstructed image. However, the results depends on how the mean-square-error is defined if the amplitude errors are not small. An appropriate approach appears to be to calculate the mean-square-error based on reconstructed images that are scaled by energy to the original image.

Scaling tends not to affect the contrast in the image, which is one of the primary attributes to which the visual system responds. Therefore, a mean-square-error minimised over contrast-independent scaling would appear to be an appropriate metric to compare images in terms of visual interpretability. This is borne out by the images in the rows of Figs. 5 and 6 having similar visual quality, i.e. the energy-normalised metrics  $\langle e_s^2 \rangle$  and  $\langle e_z^2 \rangle$  appear to track, in general terms, the perceived quality of the images. Of course when one considers specific image features, such as edges, to which the visual system is particularly sensitive, the mse is not a particularly appropriate metric.

The error  $\langle e_s^2 \rangle$  allows an analytical expression for the image error in terms of the amplitude and phase errors to be derived if the effect of saturation of negative amplitudes is ignored. Saturation of negative amplitudes is important for larger amplitude errors however, and the relative effects of spectral amplitude and phase errors are evaluated by simulations (Fig. 6). The results indicate that for large errors, both amplitude and phase errors destroy the interpretability of reconstructed images. The effects of small errors and are obviously less severe, and the effects of amplitude and phase errors appear to be similar although further study is needed in this case.

The similarly poor quality of the images in the bottom row of Fig. 6 may appear to be at odds with phase dominance. These images correspond to almost random amplitudes (left) and almost random phase (right). However phase dominance, in the usual sense as outlined in Section 1, is described in terms of two images. Replacing the phase of an image by the phase of another image introduces very large phase errors. However, replacing the spectral amplitude of an image by that from another image (which has a similar overall distribution of spectral amplitudes, and a similar distribution with spatial frequency) does not introduce errors that are as severe as results from replacing the amplitudes by random values.

There are still some interesting open questions on this topic. One is, what are the more detailed effects of more modest amplitude and phase errors? This could be answered by a similar study with more finely resolved values of  $\langle e_z^2 \rangle$ . Another is, what are the effects if the amplitude errors are such that the resulting amplitudes track the overall distribution of amplitudes with spatial frequency as in the original image?

## 7 Acknowledgements

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