Reconstruction Formulas for Rotational Dynamic Stereo

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Abstract

3D reconstruction is one of the main components in computer vision. The dynamic stereo model applied to rotating objects on a turntable provides a way of analyzing object's surface or 3D position. In this paper, we extend work reported in [5]: we correct an error in computing the rotational angle; and we present a new procedure for 3D object reconstruction with unknown rotation angle by using orthogonal coordinates in a dynamic stereo model.

Keywords: 3D reconstruction, dynamic stereo, turntable, rotational objects.

1 Introduction

The dynamic stereo model can be used to reconstruct surfaces of objects or to analyze 3D positions, see Fig. 1. We consider an object on a turntable, images are captured at different times. See [1] page 209 Task 5.2: For a surface point $P = (X_w, Y_w, Z_w)$, it is assumed that its motion from image E_i to image $E_i + 1$ could be tracked exactly. The rotation angle θ of the turntable is assumed to be known. The world coordinates X_w, Y_w and Z_w of the point P have to be determined. This is to analyze the 3D position of point P. Actually the rotation angle θ is not necessary to be known. [5] presented two solutions for calculating the rotation angle θ. One is a straightforward solution, and the other is using cylinder coordinates. We found that the formula on calculating angle θ is not correct in the straightforward solution. In Sec. 3 we will correct it and show our detailed solution. Besides the calculation of angle θ using cylinder coordinates, we also present our solution of calculating angle θ using orthogonal coordinates.

The paper is structured as follows: Section 2 presents our method for calculating the rotation angle θ using orthogonal coordinates; Section 3 presents our calculation procedure for correcting the formula stated in [5]; and Section 4 gives our conclusions.

2 Orthogonal Coordinates

[5] presented an approach for calculating 3D object positions by using cylinder coordinates. It defined an upward Z direction and used a left-hand coordinate system. In this section, we present an alternative approach by using orthogonal coordinates and the right-hand coordinate system for reconstructing 3D objects. Assume a pair of points in image coordinates (x_1, y_1) and (x_2, y_2), the corresponding world coordinate points are $P = (X,Y,Z)$ and the point after rotation is $P' = (X',Y',Z')$. The corresponding camera coordinate points are $C = (X_k, Y_k, Z_k)$ and the point after rotation is $C' = (X'_k, Y'_k, Z'_k)$. The distorted image coordinates (x_{v_1}, y_{v_1}) and (x_{v_2}, y_{v_2}) can be calculated from

$$\begin{bmatrix} x_b \\ y_b \end{bmatrix} = \begin{bmatrix} \frac{s_x x_v}{d'_x} \\ \frac{y_v}{d_y} \end{bmatrix} + \begin{bmatrix} c_x \\ c_y \end{bmatrix}.$$

That is,

$$x_{v_1} = \frac{x_1 d'_x - c_x}{s_x} \text{ and } y_{v_1} = y_1 d_y - c_y,$$

$$x_{v_2} = \frac{x_2 d'_x - c_x}{s_x} \text{ and } y_{v_2} = y_2 d_y - c_y.$$

The undistorted image coordinates (x_{u_1}, y_{u_1}) and (x_{u_2}, y_{u_2}) can be calculated from

$$\begin{bmatrix} x_v \\ y_v \end{bmatrix} = \begin{bmatrix} x_u \\ y_u \end{bmatrix} - \begin{bmatrix} D_x \\ D_y \end{bmatrix},$$

where

$D_x = x_v(k_1 r^2 + k_2 r^4)$ and $D_y = y_v(k_1 r^2 + k_2 r^4))$

with $r = \sqrt{x_v^2 + y_v^2}$. That is,

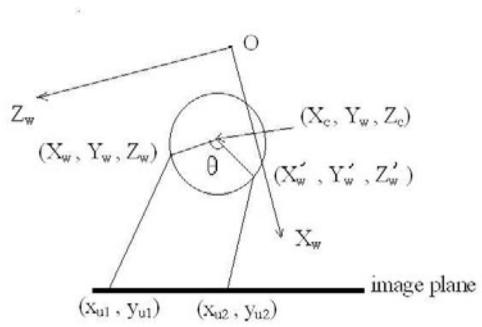

Figure 1: Dynamic stereo model (Y axis towards viewer).

$x_{u_1} = x_{v_1} + D_{x_1}$ and $y_{u_1} = y_{v_1} + D_{y_1}$,

$x_{u_2} = x_{v_2} + D_{x_2}$ and $y_{u_2} = y_{v_2} + D_{y_2}$,

The undistorted image coordinates can be obtained from the corresponding camera coordinates by using central projection,

$$x_u = \frac{fX_k}{Z_k} \text{ and } y_u = \frac{fY_k}{Z_k}.$$

Therefore, the camera coordinates can be rewritten as in the following form:

$$C = Z_k \begin{bmatrix} \frac{x_{u_1}}{f} \\ \frac{y_{u_1}}{f} \\ 1 \end{bmatrix} = Z_k \cdot E \quad \text{and} \tag{1}$$

$$C' = Z_k' \begin{bmatrix} \frac{x_{u_2}}{f} \\ \frac{y_{u_2}}{f} \\ 1 \end{bmatrix} = Z_k' \cdot E'. \tag{2}$$

Consider the affine transformation from world into camera coordinates, with rotation matrix R and translation vector T,

$$R = \begin{bmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \\ r_7 & r_8 & r_9 \end{bmatrix} \text{ and } T = \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix}.$$

The world coordinates can be transformed to the camera coordinates by using the following equation

$$C = R \cdot P + T \text{ and } C' = R \cdot P' + T.$$

From equations (1) and (2), we have

$$P = R^{-1}(Z_k \cdot E - T) = R^T(Z_k \cdot E - T) \text{ and} \tag{3}$$

$$P' = R^{-1}(Z_k' \cdot E' - T) = R^T(Z_k' \cdot E' - T). \tag{4}$$

Assumption 1. The rotation axis is parallel to the Y axis of the world coordinate system (see Fig. 1).

If the dynamic stereo system satisfies assumption 1, the rotation axis is across the point $T_c = (X_c, 0, Z_c)$. The values of X_c and Z_c can be calibrated. The point $P' = (X_w', Y_w', Z_w')$ is the transformation point of $P = (X_w, Y_w, Z_w)$ which is rotated θ degree around the rotation axis.

Assume that our world coordinate system is a right-hand system, as already stated in [1] the rotation matrix R_θ is as follow

$$R_\theta = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}.$$

The world coordinate points P' and P satisfy the following equation

$$P' = T_c + R_\theta \cdot (P - T_c).$$

From equations (3) and (4), the above equation can be rewritten as (see [1])

$$R^T \cdot (Z_k' \cdot E' - T) = T_c + R_\theta \cdot (R^T \cdot (Z_k \cdot E - T) - T_c). \tag{5}$$

Although the algorithm still assumes that the rotation angle is given (see [1, page 209]), it is known that in general the rotation angle θ can be determined from one pair of corresponding projections of the same surface point [5]. The system $\mathbf{A} \cdot \mathbf{z} = \mathbf{b}$ of linear equations allows that the rotation angle θ does not have to be assumed to be known (see [1, page 212]). We will prove that the rotation angle is unnecessary to be known. In order to prove this, first we need to compute Z_k. Then it follows that Z_k' can be calculated, and our theorem can be proved. The details of calculating Z_k and Z_k' are in Appendix A.

Theorem 2.1 *In equations (1) and (2), assume that* $Z_k \neq 0$; $Z_k' \neq 0$ *and* $a_2' \neq 0$; *and* $a_1^2 + a_3^2 - k'^2(a_1'^2 + a_3'^2) \neq 0$. *Then we have*

$$Z_k = \frac{2[-(a_1b_1 + a_3b_3) + k'(a_1'b_1 + a_3'b_3)]}{(a_1^2 + a_3^2) - k'^2(a_1'^2 + a_3'^2)}$$

and

$$Z_k' = k'Z_k.$$

For this theorem, we use

$e_1 = \frac{x_{u_1}}{f}$ and $e_2 = \frac{y_{u_1}}{f}$,

$e'_1 = \frac{x_{u_2}}{f}$ and $e'_2 = \frac{y_{u_2}}{f}$,

$a_1 = r_1e_1 + r_4e_2 + r_7, \quad a_2 = r_2e_1 + r_5e_2 + r_8,$

$a_3 = r_3e_1 + r_6e_2 + r_9, \quad a'_1 = r_1e'_1 + r_4e'_2 + r_7,$

$a'_2 = r_2e'_1 + r_5e'_2 + r_8, \quad a'_3 = r_3e'_1 + r_6e'_2 + r_9,$

$b_1 = b'_1 = -(r_1T_x + r_4T_y + r_7T_z + X_c),$

$b_2 = b'_2 = -(r_2T_x + r_5T_y + r_8T_z + 0),$

$b_3 = b'_3 = -(r_3T_x + r_6T_y + r_8T_z + Z_c)$

and

$k' = \frac{a_2}{a'_2}$

provided that

$a'_2 \neq 0.$

3 3D Position from Unknown Angle

[5] discusses in section "World Position for Unknown Rotation" a straightforward solution. However, equation (29) of calculating rotation θ is incorrect. In this section, we will correct this formula and present a detailed derivation. Compared with [5], we will use symbols a_1, a_2, a_3 instead of a_x, a_y, a_z, same changes to b and c. So we also have b_1, b_2, b_3 and c_1, c_2, c_3. From equations (25), (26) and (27) of page 132 in [5] we can get the following formula:

$$z_1a_1 - c_1 = (z_2b_1 - c_1)\cos\theta - (z_2b_2 - c_2)\sin\theta \quad (6)$$

$$z_1a_2 - c_2 = (z_2b_1 - c_1)\sin\theta + (z_2b_2 - c_2)\cos\theta \quad (7)$$

$$z_1a_3 - c_3 = z_2b_3 - c_3 \quad (8)$$

Solve z_2 from equation (8), and use it in equation (6) and (7), it follows that:

$z_1a_1 - c_1$

$=(\frac{z_1a_3}{b_3}b_1 - c_1)\cos\theta - (\frac{z_1a_3}{b_3}b_2 - c_2)\sin\theta$

$= z_1(\frac{a_3}{b_3}b_1\cos\theta - \frac{a_3}{b_3}b_2\sin\theta) - c_1\cos\theta + c_2\sin\theta$

(9)

By (9), we have

$$\begin{aligned} z_1 &= \frac{c_1 - c_1\cos\theta + c_2\sin\theta}{a_1 - \frac{a_3}{b_3}b_1\cos\theta + \frac{a_3}{b_3}b_2\sin\theta} \\ &= \frac{b_3(c_1 - c_1\cos\theta + c_2\sin\theta)}{a_1b_3 - a_3b_1\cos\theta + a_3b_2\sin\theta} \end{aligned} \quad (10)$$

Similarly,

$$z_1 = \frac{b_3(c_2 - c_1\sin\theta - c_2\cos\theta)}{a_2b_3 - a_3b_1\sin\theta - a_3b_2\cos\theta} \quad (11)$$

Compare (10) with (11), we get

$$\begin{aligned} z_1 &= \frac{b_3(c_1 - c_1\cos\theta + c_2\sin\theta)}{a_1b_3 - a_3b_1\cos\theta + a_3b_2\sin\theta} \\ &= \frac{b_3(c_2 - c_1\sin\theta - c_2\cos\theta)}{a_2b_3 - a_3b_1\sin\theta - a_3b_2\cos\theta} \end{aligned} \quad (12)$$

or

$$\begin{aligned} z_1 &= \frac{c_1 - c_1\cos\theta + c_2\sin\theta}{a_1b_3 - a_3b_1\cos\theta + a_3b_2\sin\theta} \\ &= \frac{c_2 - c_1\sin\theta - c_2\cos\theta}{a_2b_3 - a_3b_1\sin\theta - a_3b_2\cos\theta} \end{aligned} \quad (13)$$

Note that

$$\sin\theta = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \frac{2x}{1+x^2} \quad (14)$$

and

$$\cos\theta = \frac{1-x^2}{1+x^2} \quad (15)$$

where

$$x = \tan\frac{\theta}{2} \quad (16)$$

After replacing $\sin\theta$ and $\cos\theta$ with (14) and (15), (13) becomes

$$\begin{aligned} z_1 &= \frac{c_1 - c_1\frac{1-x^2}{1+x^2} + c_2\frac{2x}{1+x^2}}{a_1b_3 - a_3b_1\frac{1-x^2}{1+x^2} + a_3b_2\frac{2x}{1+x^2}} \\ &= \frac{c_2 - c_1\frac{2x}{1+x^2} - c_2\frac{1-x^2}{1+x^2}}{a_2b_3 - a_3b_1\frac{2x}{1+x^2} - a_3b_2\frac{1-x^2}{1+x^2}} \end{aligned} \quad (17)$$

So (17) can be reduced to

$$\begin{aligned}
&[c_1(a_2b_3+a_3b_2)-c_2(a_1b_3+a_3b_1)]x^3 \\
&+[-2a_3(b_1c_1-b_2c_2)+c_2(a_2b_3+a_3b_2) \\
&\qquad +c_1(a_1b_3+a_3b_1)]x^2 \\
&+[-2a_3(b_1c_2-b_2c_1)+c_1(a_2b_3-a_3b_2) \\
&\qquad -c_2(a_1b_3-a_3b_1)]x \\
&+[c_2(a_2b_3-a_3b_2)+c_1(a_1b_3-a_3b_1)]=0 \qquad (18)
\end{aligned}$$

Assume that

$$c_1(a_2b_3+a_3b_2)-c_2(a_1b_3+a_3b_1)\neq 0$$

Then equation (18) has a real root (see [6], [7], [8])

$$x=\sqrt[3]{\sqrt{\frac{b^2}{4}+\frac{a^3}{27}}-\frac{b}{2}}-\sqrt[3]{\sqrt{\frac{b^2}{4}+\frac{a^3}{27}}+\frac{b}{2}}-\frac{f}{3} \qquad (19)$$

where

$$a=g-\frac{f^2}{3}, \quad b=\frac{2f^3}{27}-\frac{fg}{3}+h$$

$$\begin{aligned}
f \;=\; & \frac{-2a_3(b_1c_1-b_2c_2)}{c_1(a_2b_3+a_3b_2)-c_2(a_1b_3+a_3b_1)} \\
& +\frac{c_2(a_2b_3+a_3b_2)}{c_1(a_2b_3+a_3b_2)-c_2(a_1b_3+a_3b_1)} \\
& +\frac{c_1(a_1b_3+a_3b_1)}{c_1(a_2b_3+a_3b_2)-c_2(a_1b_3+a_3b_1)}
\end{aligned}$$

$$\begin{aligned}
g \;=\; & \frac{-2a_3(b_1c_2-b_2c_1)}{c_1(a_2b_3+a_3b_2)-c_2(a_1b_3+a_3b_1)} \\
& +\frac{c_1(a_2b_3-a_3b_2)}{c_1(a_2b_3+a_3b_2)-c_2(a_1b_3+a_3b_1)} \\
& +\frac{-c_2(a_1b_3-a_3b_1)}{c_1(a_2b_3+a_3b_2)-c_2(a_1b_3+a_3b_1)}
\end{aligned}$$

$$h=\frac{c_2(a_2b_3-a_3b_2)+c_1(a_1b_3-a_3b_1)}{c_1(a_2b_3+a_3b_2)-c_2(a_1b_3+a_3b_1)}$$

By (16) and (19) we finally obtain

$$\theta=2\arctan x.$$

4 Conclusions

The rotational dynamic stereo model is a common model in 3D object reconstruction and object's position analysis in world coordinates. We proved that the rotation angle is not necessary to be known. This result is important, for the algorithm on page 209 in [1], which assumes that the rotation angle is known. We provided details for calculating rotation angle. Our results are illustrated in Fig.2. Here a golf head was reconstructed by placing it on a turntable.

Figure 2: Left is the original image of a golf head. Right is the reconstructed result in 3D.

References

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A Proof of Theorem 2.1.

From equation (5), we have

$$R^T\cdot(Z'_k\cdot E'-T)-T_c=R_\theta\cdot(R^T\cdot(Z_k\cdot E-T)-T_c) \quad (20)$$

The left hand side of equation (20) equals to

$$\begin{bmatrix} r_1 & r_4 & r_7 \\ r_2 & r_5 & r_8 \\ r_3 & r_6 & r_9 \end{bmatrix}\left(Z'_k\begin{bmatrix} \frac{x_{u_2}}{f} \\ \frac{y_{u_2}}{f} \\ 1 \end{bmatrix}-\begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix}\right)-\begin{bmatrix} X_c \\ 0 \\ Z_c \end{bmatrix}$$

$$= \begin{bmatrix} r_1 & r_4 & r_7 \\ r_2 & r_5 & r_8 \\ r_3 & r_6 & r_9 \end{bmatrix} \left(Z'_k \begin{bmatrix} e'_1 \\ e'_2 \\ 1 \end{bmatrix} - \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} \right) - \begin{bmatrix} X_c \\ 0 \\ Z_c \end{bmatrix}$$

$$= \begin{bmatrix} a'_1 Z'_k + b'_1 \\ a'_2 Z'_k + b'_2 \\ a'_3 Z'_k + b'_3 \end{bmatrix}. \tag{21}$$

The right hand side of equation (20) equals to

$$R_\theta \begin{bmatrix} r_1 & r_4 & r_7 \\ r_2 & r_5 & r_8 \\ r_3 & r_6 & r_9 \end{bmatrix} \left(Z_k \begin{bmatrix} \frac{x_{u_1}}{f} \\ \frac{y_{u_1}}{f} \\ 1 \end{bmatrix} - \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} \right) - \mathrm{R}_\theta \begin{bmatrix} X_c \\ 0 \\ Z_c \end{bmatrix}$$

$$= R_\theta \begin{bmatrix} r_1 & r_4 & r_7 \\ r_2 & r_5 & r_8 \\ r_3 & r_6 & r_9 \end{bmatrix} \left(Z_k \begin{bmatrix} e_1 \\ e_2 \\ 1 \end{bmatrix} - \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} \right) - \mathrm{R}_\theta \begin{bmatrix} X_c \\ 0 \\ Z_c \end{bmatrix}$$

$$= R_\theta \begin{bmatrix} a_1 Z_k + b_1 \\ a_2 Z_k + b_2 \\ a_3 Z_k + b_3 \end{bmatrix}. \tag{22}$$

From equation (22) it follows that the right hand side of equation (20) equals to

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} a_1 Z_k + b_1 \\ a_2 Z_k + b_2 \\ a_3 Z_k + b_3 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cdot(a_1 Z_k + b_1) + \\ \sin\theta\cdot(a_3 Z_k + b_3) \\ a_2 Z_k + b_2 \\ (-\sin\theta)\cdot(a_1 Z_k + b_1) + \\ \cos\theta\cdot(a_3 Z_k + b_3) \end{bmatrix}. \tag{23}$$

From (21) and (23), we have

$$\cos\theta\cdot(a_1 Z_k + b_1) + \sin\theta\cdot(a_3 Z_k + b_3) = a'_1 Z'_k + b'_1, \tag{24}$$

$$a_2 Z_k + b_2 \quad = \quad a'_2 Z'_k + b'_2, \text{ and} \tag{25}$$

$$(-\sin\theta)\cdot(a_1 Z_k + b_1) + \cos\theta\cdot(a_3 Z_k + b_3) = a'_3 Z'_k + b'_3. \tag{26}$$

From equation (25) and note that $b_2 = b'_2$, we obtain

$$a_2 Z_k = a'_2 Z'_k. \tag{27}$$

From equation (10), it follows that

$$a'_1 Z'_k + b'_1 = \cos\theta\cdot(a_1 Z_k + b_1) + \sin\theta\cdot(a_3 Z_k + b_3). \tag{28}$$

From equation (24), it follows that

$$a'_3 Z'_k + b'_3 = (-\sin\theta)\cdot(a_1 Z_k + b_1) + \cos\theta\cdot(a_3 Z_k + b_3). \tag{29}$$

Squaring equations (28) and (29), we have

$(a'_1 Z'_k + b'_1)^2 + (a'_3 Z'_k + b'_3)^2$

= $[\cos\theta\cdot(a_1 Z_k + b_1) + \sin\theta\cdot(a_3 Z_k + b_3)]^2$

+$[$(-sin$\theta)\cdot(a_1 Z_k + b_1) + \cos\theta\cdot(a_3 Z_k + b_3)]^2$

(30)

Now the left hand side of equation (30) equals to

$$(a'^2_1 + a'^2_3) Z'^2_k + 2(a'_1 b'_1 + a'_3 b'_3) Z'_k + (b'^2_1 + b'^2_3)$$

Since

$b'_1 = b_1$ and $b'_3 = b_3$,

then the left hand side of equation (31) equals to

$$(a'^2_1 + a'^2_3) Z'^2_k + 2(a'_1 b'_1 + a'_3 b'_3) Z'_k + (b^2_1 + b^2_3) \tag{31}$$

And the right hand side of (31) equals to

$$\begin{aligned} = \quad & (a_1 Z_k + b_1)^2(\cos^2\theta + \sin^2\theta) \\ & + (a_3 Z_k + b_3)^2(\sin^2\theta + \cos^2\theta) \\ & + 2\cos\theta\cdot(a_1 Z_k + b_1)\cdot\sin\theta\cdot(a_3 Z_k + b_3) \\ & - 2\cos\theta\cdot(a_1 Z_k + b_1)\cdot\sin\theta\cdot(a_3 Z_k + b_3) \\ = \quad & (a^2_1 + a^2_3) Z^2_k + 2(a_1 b_1 + a_3 b_3) Z_k + (b^2_1 + b^2_3) \end{aligned} \tag{32}$$

From (31) and (32), we have

$$\begin{aligned} & (a'^2_1 + a'^2_3) Z'^2_k + 2(a'_1 b'_1 + a'_3 b'_3) Z'_k \\ = & (a^2_1 + a^2_3) Z^2_k + 2(a_1 b_1 + a_3 b_3) Z_k. \end{aligned} \tag{33}$$

We combine equation (27) with the second condition in the theorem. It follows that

$$\begin{aligned} Z'_k &= \frac{a_2}{a'_2} Z_k \\ &= k' Z_k, \text{ where } k' = \frac{a_2}{a'_2}. \end{aligned} \tag{34}$$

From equation (33), we have

$$[(a_1^2 + a_3^2) - k'^2(a_1'^2 + a_3'^2)]Z_k^2 +$$

$$2[(\mathrm{a}_1 b_1 + a_3 b_3) - k'(a'_1 b_1 + a'_3 b_3)]Z_k = 0.$$

From the first condition in the theorem it follows that

$$Z_k \neq 0.$$

In accordance with the third condition in the theorem:

$$a_1^2 + a_3^2 - k'^2(a_1'^2 + a_3'^2) \neq 0.$$

So

$$Z_k = \frac{-2[(a_1 b_1 + a_3 b_3) - k'(a'_1 b_1 + a'_3 b_3)]}{(a_1^2 + a_3^2) - k'^2(a_1'^2 + a_3'^2)}.$$

Or

$$Z_k = \frac{2[-(a_1 b_1 + a_3 b_3) + k'(a'_1 b_1 + a'_3 b_3)]}{(a_1^2 + a_3^2) - k'^2(a_1'^2 + a_3'^2)}.$$

This together with equation (34), it follows that Z'_k can be calculated. Our theorem is now proved. **QED.**