3D Articulated Structure and Motion Analysis from Monocular Images

Xiaoyun Zhang,Yuncai Liu
Institute of Image Processing and Pattern Recognition,
Shanghai Jiaotong University, Shanghai, 200030, China
xiao_yun@sjtu.edu.cn, whomliu@sjtu.edu.cn

Abstract

This paper presents a new method of articulated structure and motion analysis from monocular perspective images. An articulated object is modeled as a kinematic chain consisting of joints and links, and its joint structure is estimated within a scale factor using the connection relationship of two links over two images. Then, twists and exponential maps in robotic manipulation are employed to represent the motion of each link, including the general motion of the base link and the rotation of other links around their joints. Finally, constraints from image point correspondences, which is similar to that of the essential matrix in rigid motion, are used to estimate the motion. Simulations and experiments on real images show the correctness and efficiency of the algorithm.

Keywords: Articulated object, motion estimation, kinematic chain, twist, exponential map

1 Introduction

Articulated structure and motion analysis is challenging due to its non-rigid nature, self-occlusion, variable appearance, high degree of freedom, and so on. Model based approaches are able to overcome these problems to a great extent, which have been explored by many researchers[1][2][3]. However, model acquisition and initialization are relatively under-investigated for articulated objects, especially for the case of single camera systems, which are usually performed manually by the user.

In analysis of articulated motion, it is important to automatically build articulated models from visual data directly, in which the first and most important step is to locate the joint positions of articulated objects. After the joint positions are determined, the kinematic chain of the object is obtained by connecting the joints, and a model can be fleshed out from the chain. Thus, we only need to estimate the joint angle of each link in the model based motion and pose tracking. Although commercial devices for motion capture or joint localization have been available, they are based on electromechanical or electromagnetic sensors [4] and optical retroreflective markers [5]. They are intrusive and uncomfortable for applications. Krahnstoever *et al*. [6] use 2D affine transform obtained from the motion segmentation to estimate the joint position. Iiyama *et al*. [7] extract rigid parts and the rigid motion parameters using 3D volume data under the constraint of rigid transformation, then estimates the joint points from the motion parameters. The full vision-based joint localization and estimation is still need to be investigated.

In subsequent motion tracking after modeling the observed object, most work assume that the initial state of the object is known, such as a special start pose. For example, in [8], all joint angles of the initial model are at zero degrees; Bregler and Malik [9] conduct articulated motion tracking based on an initial pose and angular configuration of the first frame. Such initialization is troublesome and often not exact. Covell *et al*. [10] conduct articulated-pose estimation using brightness and depth constancy constraints, but they assume depth information is available from either stereo cameras or other sensors.

In this paper, we propose an automatic model acquisition and initialization scheme. Using two frames from the image sequence, the joint positions of the articulated object and its initial pose are estimated. With a model of kinematic chain in robotics, we describe the motion of an articulated object by twists and exponential maps [9]. Although twists and exponential maps have been used in [9], it relates the image motion with 3D model motion and is a differential method suitable for very small motion between consecutive images. But our work can deal with large frame-to-frame motion, since we use image point correspondences to constrain the motion parameters.

In our method, the initial configuration estimation and kinematic chain representation greatly decrease the number of motion parameters for subsequent images, and it becomes much easier to estimate articulated motion. Furthermore, the motion of each link is correlated through joints. This correlation demonstrates the characteristics of articulated motion, making the complexity of the problem decreased and

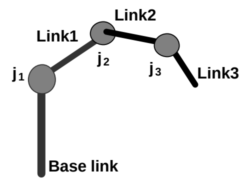

Figure 1: A kinematic chain for a basic articulated object, with a base link and three extended links.

the robustness of motion estimation improved. Our method gains a great advantage over the articulated motion estimation that first partition the object into rigid components and then estimate the motion of each subpart respectively, which ignores the very useful information provided by the interdependencies of articulated components. It is also easy and cheap since we use a single camera system and no sensors are required.

2 Articulated Object Model and System Settings

Generally, articulated objects consist of kinematic chains, connected to a base link. In human models, we may regard the torso as the base link and four limbs as connected kinematic chains[3][11]. In hand model, the palm is regarded as base link and its five fingers as the extended kinematic chains [8]. In [12], a connecting tree with a torso as foot node is used to describe the structural knowledge about human body. A person is represented by a cardboard model consisting of a set of connected planar patches in [13]. Therefore, in this paper we will mainly focus on the joint positions and motion estimation of an open kinematic chain such as that shown in Fig.1.

One DOF revolute joint is the simplest joint between two connected links, which allows rotation motion about a single axis. Because of its simplicity of mechanical structure, revolute joint is the joint most commonly used in robotic manipulation. A combination of two or three revolute joints with different axes of rotation can model high DOF joints. In this paper, we will concentrate on the one DOF revolute joints.

We suppose that a stationary pinhole camera is observing a moving articulated object. The 3D world coordinate system is chosen to be fixed on the camera with the origin coinciding with the center of the camera and the z-axis coinciding with the optical axis and pointing to the front of the camera. The image plane is located at $z = f$, the focal length of the camera, with its coordinates axes X and Y parallel to the axes x and y of the 3D world coordinates, respectively. Then according to perspective projection, we have $p = \frac{z}{f}P$, i.e.,

$$(x,y,z)^T = \frac{z}{f}(X,Y,f)^T \tag{1}$$

where (x,y,z) is the world coordinates of a 3D point p, and (X,Y) is the coordinates of its correspondent image point P in the image plane.

3 Articulated Motion Equations Over Two Frames

For an articulated object with two links, we call the first link as base link, denoted by $link_0$, and the other link by $link_1$; the joint j_1 between $link_0$ and $link_1$ is a one DOF revolute joint. Since the two links are connected together by a joint, their motion are not independent. We explain their motion as follows: first, $link_0$ and $link_1$ rotate around the joint j_1 to have their respective after-motion orientations, and then translate together to the destination position. As a result, the joint j_1 which is on both $link_0$ and $link_1$ satisfies the motion of both links. We have the following motion equations,

$$p'_0 = R_0(p_0 - j_1) + j'_1 \tag{2}$$
$$p'_1 = R_1(p_1 - j_1) + j'_1 \tag{3}$$

where R_0, R_1 are rotation matrices of $link_0$ and $link_1$ respectively, $p_0 \leftrightarrow p'_0$ is point correspondence on $link_0$, $p_1 \leftrightarrow p'_1$ is point correspondence on $link_1$. We should be clear that R_1 is the rotation of $link_1$ in the world coordinates, not the rotation relative to $link_0$. From (2)(3), we see that the motions of the two links are correlated with each other by the joint. This correlation plays an important role in articulated structure and motion analysis, and will be used to estimate the joint in our work.

4 Initial Configuration Estimation

In the following, we mainly consider the estimation of the joint between $link_0$ and $link_1$. For the joint between $link_i$ and $link_{i+1}$ $(i = 1,\cdots,N)$, we can deal with in a similar way.

4.1 Constraints from Image Point Correspondences

In a similar way to derive the well-known essential matrix [14] equation in rigid motion estimation, pre-crossing both sides of Eq.(2) by $j'_1 - R_0 j_1$ and then pre-multiplying by $p_0'^T$, we can derive

$$p_0'^T[(j'_1 - R_0 j_1) \times R_0 p_0] = 0 \tag{4}$$

In perspective projection, 3D point p and its correspondent image P have the relationship of (1). Therefore, we derive an essential constraint equation about the motion and joint parameters

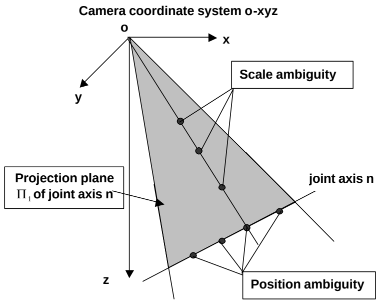

Figure 2: Ambiguities of joint position

$$P_0'^T E_0 P_0 = 0 \tag{5}$$

where

$$E_0 = [j_1']_\times R_0 - R_0 [j_1]_\times \tag{6}$$

and $[\cdot]_\times$ is a mapping from a 3-dimensional vector to a 3 by 3 skew symmetric matrix

$$[(x,y,z)^T]_\times = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \tag{7}$$

Similarly, we can get the constraint on motion parameters for $link_1$

$$P_1'^T E_1 P_1 = 0 \tag{8}$$

with $E_1 = [j_1']_\times R_1 - R_1 [j_1]_\times$. Therefore, each pair of image point correspondence provides a constraint of (5) or (8). From a set of image point correspondences, we can get a set of such equations. Solving these equations, we might expect to estimate the motion parameters R_0, R_1 over the two image frames and the joint position j_1 between $link_0$ and $link_1$. However, for one DOF revolute joint, the solution for j_1 satisfying Eqs.(2) is not unique. In order to get a solution of the joint position from Eqs.(5) and (8), more constraints should be enforced.

4.2 Constraints on Joint Position

It is easy to see that any point lying on the joint axis satisfies Eqs.(2)(5)(8) because of the one DOF attribute of the revolute joint. We call this position ambiguity of revolute joint. It should also be noticed that the depth information of an object is lost in monocular images. A 3D point p and a point cp have the same image in perspective projection if c is a positive constant. Therefore we can only recover the structure of an object from images up to a scale factor to its ground truth. It can be seen from Eq.(5) or (8) that the equation keeps unchanged if j_1 and j_1' are multiplied by a same factor. We call this scale ambiguity.

The position ambiguity and scale ambiguity result in that any point lying on the projection plane Π_1 of the joint axis can be a solution of Eqs.(5) and (8), see Fig.2. To avoid the scale ambiguity, we set the depth of the joint point j_1 as a positive value (the object is in front of the camera)

$$j_{1z} = z_0 \tag{9}$$

In practical application, we may set z_0 as the approximate value of the distance of the object away from the camera.

We see that even if we have set the z coordinate of j_1 as a constant, any point on the intersection line between the projection plane Π_1 and the plane of $z = z_0$ can be the joint point. So we put another constraint on j_1

$$j_{1x}^2 + j_{1y}^2 = c \tag{10}$$

where c is a pre-set positive constant. In theory, c in (10) can be set as any positive value which is larger than the shortest distance between the z axis and the intersection line between Π_1 and the plane of $z = z_0$. Otherwise, when c is set too small, there is no point on the plane Π_1 which satisfies the conditions (9) and (10).

4.3 Degeneration of the Estimation

Eq.(5) and Eq.(8) are homogeneous equations about E_0 and E_1. If $E_0 = 0$ and $E_1 = 0$, Eqs.(5) and (8) become identical equations, which should be avoided. When $E_0 = 0$, we can get $j_1' = R_0 j_1$ from its definition formula (6), leading to $j_1' \times R_0 j_1 = 0$. Therefore, in order to avoid this degeneration, we should put a constraint of

$$j_1' \times R_0 j_1 \neq 0 \tag{11}$$

Constraint (11) is difficult to deal with in numerical optimization. So, we change the inequality constraint (11) as the following equality one

$$1.0 - \frac{t^2}{\sigma^2 + t^2} = 0 \tag{12}$$

where $t = |j_1' \times R_0 j_1|$. We find that the function

$$f(x) = 1.0 - \frac{x^2}{\sigma^2 + x^2} \tag{13}$$

gets the maximum when $x = 0$ and is almost zero when x is much larger than σ. If we set the parameter σ very small, $f(x)$ likes an impulse function, but it is a continuous, differentiable function. Similarly, we use the function (13) to avoid the degeneration case of $E_1 = 0$.

4.4 Estimate Joint Position

Now, we can estimate the joint position j_1 and the rotation matrices R_0, R_1 over two image views by solving the following system of equations:

$$\begin{cases} P_{0i}'^T([j_1']_\times R_0 - R_0[j_1]_\times)P_{0i} = 0 \\ P_{1i}'^T([j_1']_\times R_1 - R_1[j_1]_\times)P_{1i} = 0 \\ j_{1z} - z_0 = 0 \\ j_{1x}^2 + j_{1y}^2 - c = 0 \\ 1.0 - \frac{|j_1' \times R_0 j_1|^2}{\sigma^2 + |j_1' \times R_0 j_1|^2} = 0 \\ 1.0 - \frac{|j_1' \times R_1 j_1|^2}{\sigma^2 + |j_1' \times R_1 j_1|^2} = 0 \end{cases} \tag{14}$$

where $i = 1, \cdots, m$ and m is the number of feature points on each link. Since each feature point correspondence provides one constraint about the joint and motion parameters, theoretically we need five feature points on each link to solve the problem, i.e., $m = 5$. But, in order to decrease the probability of converging to a local minimum and make the algorithm be robust, we use more available points in practical application.

4.5 Estimate Joint Axis

The direction of joint axis $\vec{n}_1$ of $link_1$, still denoted by $\vec{n}_1$, can be computed from the rotation matrices R_0 and R_1 already obtained. In fact, $\vec{n}_1$ is the rotation axis of $link_1$ when the base link $link_0$ keeps stationary. We can explain the articulated motion of the two links in the following way: firstly $link_0$ moves, with $link_1$ moving rigidly together with it; then $link_0$ keeps still and $link_1$ makes a rotation around the joint of $link_1$ to reach its final position. In terms of R_0, R_1, the motion can be considered as that the joint axis $\vec{n}_1$ is first transferred to $R_0\vec{n}_1$ after $link_0$'s motion, then $link_1$ carries out a relative rotation $R_{10} = R_1 R_0^{-1}$ around the joint axis $R_0\vec{n}_1$. So, the rotation axis of rotation matrix R_{10} is $R_0\vec{n}_1$. As a result, we can compute the joint axis $\vec{n}_1$ of $link_1$ as follows

$$\vec{n}_1 = R_0^{-1} axis(R_{10}) \tag{15}$$

where $axis(R)$ means the direction of rotation axis of matrix R, which can be easily computed from R.

4.6 Scale Consistency

Using the similar way of estimating the joint of $link_1$, we can estimate the joint position j_i and axis $\vec{n}_i$ of $link_i (i = 2, \cdots, N)$. Each estimated joint position j_i has been determined upto a scale factor to its correspondent ground truth, since the absolute depth information is lost in monocular images. However, all the estimated joints should have the same scale factor to their ground truth because of the articulation constraints.

Since j_2 is also a point on $link_1$, it satisfies the motion of $link_1$, i.e., we have

$$j_2' = R_1(j_2 - j_1) + j_1' \tag{16}$$

Therefore, for the estimated joint points $\hat{j}_1, \hat{j}_1'$ and $\hat{j}_2, \hat{j}_2'$, if

$$s|\hat{j}_2' - R_1\hat{j}_2| = |\hat{j}_1 - R_1\hat{j}_1| \tag{17}$$

scale $\hat{j}_2$ by s: $\hat{j}_2 \Leftarrow s\hat{j}_2$, then $\hat{j}_2$ will have a same scale as $\hat{j}_1$ to the joints of the ground truth. We can scale the other joint point j_i in a similar way.

5 Articulated Motion Estimation

5.1 Twists and Exponential Maps for Kinematic Chain

The one DOF revolute joint in kinematic chains can be represented by a twist [15]

$$\xi = \begin{bmatrix} -\vec{n} \times j \\ \vec{n} \end{bmatrix} \tag{18}$$

where $\vec{n}$ is the unit direction along the axis of the revolute joint and j is a point on the axis. Then the motion of the link, a rotation of angle θ around this joint, can be written as an exponential map

$$G = e^{\hat{\xi}\theta} \tag{19}$$

with

$$\hat{\xi} = \begin{bmatrix} [\vec{n}]_\times & -\vec{n} \times j \\ 0 & 0 \end{bmatrix} \tag{20}$$

where $\hat{\xi}$ is a 4×4 matrix with the upper 3×3 sub-matrix as a skew-symmetric matrix. The computed G must be a 4×4 matrix in the form of

$$G = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix} \tag{21}$$

with its upper 3×3 sub-matrix R as a rotation matrix and T is a 3D vector. Thus G is in fact a rigid body motion transformation in $\Re^3$ using homogeneous coordinates, describing the motion of the link in world coordinates when other links keep stationary.

The motion of each link except the base link can be represented by its rotation angle. Suppose the motion of the base link $link_0$ is expressed by the transformation G_0, and the motion of $link_1$ relative to $link_0$ is $e^{\hat{\xi}_1\theta_1}$. The twist ξ_1 can be computed from the estimated joint axis $\vec{n}_1$ and the joint position j_1, θ_1 is the to be solved rotation angle. Then the overall motion G_1 of $link_1$ is a composite of the motion of the base link and its motion relative to $link_0$

$$G_1 = G_0 e^{\hat{\xi}_1 \theta_1} \tag{22}$$

Similarly, the motion transformation G_i of $link_i$ is the product of exponential maps

$$G_i = G_0 e^{\hat{\xi}_1 \theta_1} \cdots e^{\hat{\xi}_i \theta_i} \tag{23}$$

$e^{\hat{\xi}_i \theta_i}$ is the motion of $link_i$ relative to its previous connected link $link_{i-1}$, which is in fact the motion transformation of $link_i$ when all the other links are kept stationary. The product of exponential maps, G_i, must have the same form as G in (21).

5.2 Estimation of Articulated Motion

Using the motion transformation G_i of $link_i$, we have

$$\bar{p}_i' = G_i \bar{p}_i = \begin{bmatrix} R_i & T_i \\ 0 & 1 \end{bmatrix} \bar{p}_i \tag{24}$$

where $\bar{p}_i$ and $\bar{p}_i'$ are points on $link_i$ before and after motion represented in homogeneous coordinates $\bar{p}_i = (p_i^T 1)^T$. Since G_i has the form of (21), we can get the same constraint of essential matrix in rigid motion for each link, i.e.,

$$P_i'^T E_i(G_0, \theta_1, \theta_2, \cdots, \theta_i) P_i = 0 \tag{25}$$

where $P_i \leftrightarrow P_i'$ is image point correspondence on $link_i$, E_i is computed from G_i by $E_i = T_i \times R_i$. And, E_i is the function of motion parameters G_0 of $link_0$ and rotation angle $\theta_k (k = 1, \cdots, i)$ of $link_k$.

Provided with image point correspondences on each link between two images, we have the following constraints on motion parameters G_0 and θ_i

$$\begin{cases} P_0'^T E_0(G_0) P_0 = 0 \\ P_1'^T E_1(G_0, \theta_1) P_1 = 0 \\ \cdots\cdots \\ P_N'^T E_N(G_0, \theta_1, \theta_2, \cdots, \theta_N) P_N = 0 \end{cases} \tag{26}$$

There are six independent parameters in G_0 for the base link and only one θ_i for rotation angle of $link_i$,$i = 1, \cdots, N$. As a result, for an articulated object with $N+1$ components, we need only $6+N$ parameters to represent the motion, which is much less than $6(N+1)$ for the case of treating each component independently. Since each image point provide one constraint on the motion parameters, totally $6+N$ points are necessary to solve the problem in theory. If 6 points are available on the base link, only one point on each link is necessary. The hard occlusion problem in articulated motion analysis is overcome to some extent because of the requirement for a few of feature points in our work.

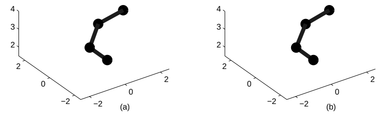

Figure 3: (a)The ground truth of the initial pose of the articulated links; (b) The estimated initial pose.

Another advantage of our work is that the motion parameters of links are interdependent. Such interdependency provides powerful information for estimating the motion parameters efficiently and robustly, because more constraints are available for each parameter. For example, the motion G_0 of $link_0$ is included in all the constraint equations of other links; the angle θ_i of $link_i$ is related with $link_k (k = i+1, \cdots, N)$. Therefore, even there are less than 6 points available on the base link, we can have enough constraints on G_0 if more points are available on other links.

6 Experiments

Levenberg-Marquardt optimization method is used to solve the equation systems involved in the algorithm.

6.1 Simulations

Suppose the feature points of three links of an articulated object are distributed randomly within the range $[-2.5, -0.8] \times [-2.5, -0.8]$, $[-0.8, 0.8] \times [-0.8, 0.8]$ and $[0.8, 2.5] \times [0.8, 2.5]$ respectively. The distance of the object from the camera ranges from 3.0 to 4.0 units. The points are projected onto an ideal image plane of 500×500 pixels with a unit focal length, corresponding to a viewing angle of about 90° and the pixel size of 0.003.

Fifteen points on each link are available for the estimation of joint structure which are corrupted with uniform distributed noise in the range of [-0.0045, 0.0045], equivalent to three pixel noise. The estimated result is illustrated in Fig.3. In the subsequent motion estimation, 8 points on each link corrupted by 5 pixels noise are used. The result is shown in Fig.4.

6.2 Experiments Using Real Scene Images

We test our algorithm by the images of a robot arm (See Fig.5), an articulated object with two links in the experiment. The images are captured by a digital camera of Nikon COOLPIX5000 with image size of 2560×1920. The camera are calibrated using the the Calibration Toolbox for Matlab, getting the focal length of 7.4251mm and the principal point of (1306.63627,981.94309) in pixels. The feature points are detected and matched manually for the moment.

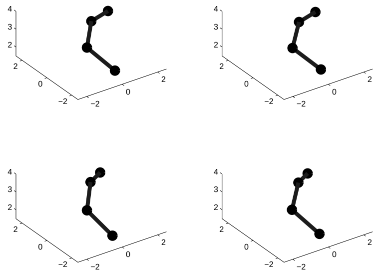

Figure 4: The articulated links of Frame 6 and 8. Left column is the ground truth pose, right column is the estimated.

Figure 5: Two images in the experiment

Table 1: The estimated initial configuration

	Estimated value	Ground truth
axis1	$(0.1858, -0.0032, -0.9735)^T$	–
point1	$(-2.0860, -0.4022, 8.6)$	–
angle1	37.5104	37.91
axis2	$(-0.0149, -0.0091, -0.9998)^T$	–
point2	(-0.0640, 1.5707, 7.5454)	–
angle2	30.3473	29.68

Seventeen and eleven feature points on link1 and link2 are respectively used to estimate the joint structure. The ground truth of the rotation angle of each link is obtained from the manipulation process of the robot.

The estimated results are shown in Table 1, where point1 and point2 are the estimated joint positions of the two links respectively. Then, the rotation angle of each link between the current frame and the initial frame is estimated using the estimated initial configuration. Thirteen and eight feature points are available on the two links respectively. The results in Fig.6 show very good performance of the algorithm, and also verify the initial joint structure estimation of the object.

7 Conclusion

In this paper, we have proposed a new method of articulated structure and motion analysis from monocular perspective images. The characteristic of articulated motion, i.e., motion correlation among links, is applied to decrease the complexity of the problem and improve the robustness.

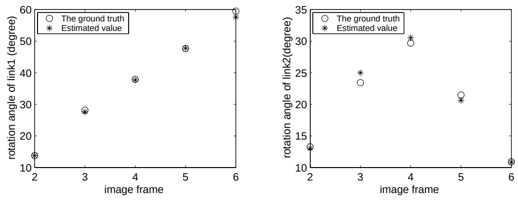

Figure 6: Rotation angles over six frames

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