A Time-Dependent LMS Algorithm for Adaptive Filtering

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Abstract:— A novel approach for the least-mean-square (LMS) estimation algorithm is proposed. The approach utilizes the conventional LMS algorithm with a time-varying convergence parameter $\mu_n$ rather than a fixed convergence parameter $\mu$. It is shown that the proposed time-varying LMS algorithm (TV-LMS) provides reduced mean-squared error and also leads to a faster convergence as compared to the conventional fixed parameter LMS algorithm. This paper presents a performance study for the proposed TV-LMS algorithm and other two main adaptive approaches: the least-mean square (LMS) algorithm and the recursive least-squares (RLS) algorithm. These algorithms have been tested for noise reduction and estimation in single-tone sinusoids and nonlinear narrow-band FM signals corrupted by additive white Gaussian noise. The study shows that the TV-LMS algorithm has a computation time close to conventional LMS algorithm with the advantages of faster convergence time and reduced mean-squared error.

Keyword:— Least-mean-square (LMS) algorithm, adaptive filters, Wiener filter, estimation.

1 Introduction

The primary feature of an adaptive algorithm is the ability to self-adjust its coefficients according to the system conditions. Accordingly, the adaptive algorithms can be applied under many signal conditions such as linear and nonlinear signal environments [4]. Adaptive algorithms have been extensively studied in the past few decades and have been widely used in many arenas including biomedical, image and speech processing, communication signal processing and many other applications [1, 2, 4].

In the communications industry, there are a lot of works that utilize the LMS and RLS algorithms for channel estimation, equalization, and demodulation [5, 6, 7]. The performance of these adaptive algorithms is highly dependent on their filter order and signal condition. Furthermore, the performance of the LMS algorithm also depends on the selected convergence parameter $\mu$. As for the RLS algorithm, it is also dependent on the parameter $\lambda$ (commonly known as “forgetting factor” or “exponential weighting factor” [2]). In this work we will only consider the version of RLS named as the ”growing window” RLS algorithm (with $\lambda = 1$) [2].

A new version of the LMS algorithm with time-varying convergence parameter is proposed in this paper. The time-varying LMS (TV-LMS) algorithm is utilizes a time-varying convergence parameter $\mu_n$ ($n$ being the sample count) with a time-decaying law. The basic idea behind this structure is the fact that the conventional LMS algorithm needs a relatively large value for the convergence parameter $\mu$ if we want to speed up the convergence of the filter coefficients to their optimal values, but a lower value for $\mu$ if we want more accurate estimation (less mean-squared error, MSE). We expect better performance if $\mu$ is adjusted to be time-dependent with a decaying law such that it has a large value at the beginning to ensure faster convergence of the coefficients to their optimal values, then, as time passes, the convergence parameter takes on smaller values for better estimation accuracy [2]. The parameter $\mu$ should reach a steady state whose value is application dependent. A general power decaying law has been proposed; however, other time-varying laws could also be applicable.

In this paper, we concentrate on noise reduction as the primary function of adaptive algorithms so as to compare their performance. We present a study of the conventional LMS algorithm, the proposed TV-LMS algorithm, and the RLS algorithm in terms of their execution time, filter order, mean-squared error (MSE) performance, and speed of convergence. The paper is organized as follows. In section II, we present the time-varying LMS (TV-LMS) algorithm and a brief description of the conventional LMS and RLS algorithms. In section III, we use MATLAB simulations to evaluate the performance of these al-
2 Adaptive Algorithms

In this section we present a brief description of the conventional LMS and RLS algorithms and propose the time-varying LMS algorithm.

2.1 The Adaptive LMS Algorithm

In the conventional adaptive LMS algorithm, the weight vector coefficients \( w(n) \) for the FIR filter are updated according to the formula \([1, 2, 4, 6, 7]\):

\[
\mathbf{w}(n) = \mathbf{w}(n-1) + \mu e(n) \mathbf{y}(n)
\]  

(1)

where \( \mathbf{w}(n) = [w_0(n) \ w_1(n) \ldots w_M(n)] \) (\( M + 1 \) being the filter length), \( \mu \) is the convergence parameter (sometimes referred to as step-size), \( e(n) = d(n) - z(n) \) is the output error (\( z(n) \) being the filter output), and \( d(n) \) is the reference signal. Note that \( z(n) = \mathbf{w}(n-1)\mathbf{y}^T(n) = \hat{x}(n) \), where \( \hat{x}(n) \) is the original signal and \( \mathbf{y}(n) = [y(n) \ y(n-1) \ldots y(n-M)] \) is the input signal to the filter.

2.2 The Time-Varying LMS Algorithm

The TV-LMS algorithm works in the same manner as the conventional LMS algorithm, except for a time-dependent convergence factor \( \mu_n \). The time-varying law depends on the kind of signals and their expected range of frequencies, hence it is application dependent. We will confine ourselves to single-tone sinusoids and narrowband FM signals. For single-tone sinusoids, there is an optimal value for the convergence parameter \( \mu \) that gives minimum MSE. This value of \( \mu \) is frequency dependent, as we will see in Section 3. If a sinusoidal input is expected with a frequency in the range \([f_1, f_2]\), then the practical choice of the steady-state value of the convergence parameter \( \mu_0 \) is that which corresponds to the frequency \( f_0 = (f_1 + f_2)/2 \). As for narrowband FM signals, we must first define the optimal \( \mu_o \) for the centre frequency or the mean frequency \( f_m \). To do so, the conventional LMS algorithm is used (with a single-tone of frequency \( f_m \)) to find the optimal value of \( \mu \) at that frequency. This optimal value \( \mu_o \) is used to update the time-varying convergence parameter \( \mu_n \) according to the following formula:

\[
\mu_n = \alpha_n \times \mu_o
\]  

(2)

where \( \alpha_n \) is a decaying factor. We will consider the following decaying law:

\[
\alpha_n = C \frac{1}{1+a n^b}
\]  

(3)

where \( C, a, b \) are positive constants that will determine the magnitude and the rate of decrease for \( \alpha_n \). According to the above law, \( C \) has to be a positive number larger than 1. When \( C = 1 \), \( \alpha_n \) will be equal to 1 and the new algorithm will be the same as the conventional LMS algorithm. An outline of the TV-LMS algorithm is shown in the following steps:

\[
z(n) = \mathbf{w}(n-1)\mathbf{y}^T(n)
\]  

(4)

\[
e(n) = d(n) - z(n)
\]  

(5)

\[
\alpha_n = C \frac{1}{1+a n^b}
\]  

(6)

\[
\mu_n = \alpha_n \times \mu_o
\]  

(7)

\[
\mathbf{w}(n) = \mathbf{w}(n-1) + \mu_n e(n) \mathbf{y}(n)
\]  

(8)

where \( T \) indicates matrix transposition.

2.3 The RLS Algorithm

As compared to the LMS algorithm, the RLS algorithm has the advantage of fast convergence, but this comes at the cost of increasing the complexity. The RLS algorithm consumes longer computation time and has a higher sensitivity to numerical instability than the LMS algorithm \([1, 2, 5]\). In this paper, we consider the RLS algorithm in the following form \([1, 2, 7]\):

\[
z(n) = \mathbf{w}(n-1)\mathbf{y}^T(n)
\]  

(9)

\[
e(n) = d(n) - z(n)
\]  

(10)

\[
k(n) = \frac{\mathbf{P}(n-1)z(n)}{\lambda + z^H(n)\mathbf{P}(n-1)z(n)}
\]  

(11)

\[
\mathbf{P}(n) = \frac{\mathbf{P}(n-1) - \mathbf{P}(n-1)z^H(n)k(n)}{\lambda}
\]  

(12)

\[
\mathbf{w}(n) = \mathbf{w}(n-1) + e(n)k(n)
\]  

(13)
where $H$ indicates the Hermitian property.

3 Simulation Results

In this section we discuss computer simulation of the above adaptive algorithms. The input signal for all algorithms has the form $y(t) = x(t) + n(t)$, $n(t)$ being white Gaussian noise with 2 dB power and $x(t)$ is the original signal. To begin with, a single-tone sinusoid signal is used as the original signal $x(t)$. This signal is of fundamental importance for the time-varying approach. A quadratic frequency-modulated (QFM) signal is then used to study the algorithms performance in a nonlinear narrow-band signal condition.

3.1 A single-tone sinusoid

We consider a single-tone signal ($A \sin(2\pi f_o t)$).

Fig. 1 shows the MSE performance of the conventional LMS algorithm with different values of $\mu$ and different input frequencies $f_o$ (SNR = 2 dB).

$$MSE = \frac{1}{N} \sum_{n=0}^{N} [x(n) - \hat{x}(n)]^2$$

where $x(n)$ is the original signal and $\hat{x}(n)$ is the filter output, which represents an estimate of the input signal. Fig. 1 shows that the performance for the conventional LMS algorithm is frequency dependent, with an optimal value of $\mu$ for each frequency. Hence, the choice of the optimal $\mu$ is application dependent.

Fig. 2 and Fig. 3 show the MSE performance of the TV-LMS and the conventional LMS algorithms with different values of $\mu$ and different filter orders. These figures show that both algorithms provide similar performance results. Their optimal $\mu_o$ is sitting at smaller region ($< 0.4 \times 10^{-3}$) and the performance is better when the filter order is larger. The results show that, for both TV-LMS and conventional LMS algorithms, a higher filter order should be used in order to provide a better MSE performance for the system. Comparing the filter order used in Fig. 2 with that in Fig. 3, it is clear that the optimal $\mu_o$ is dependent on the filter order as well as the frequency. The higher the filter order the smaller the optimal $\mu_o$ will be. The overall per-
performance of the TV-LMS algorithm is better than that of the conventional LMS algorithm.

Fig. 4 and Fig. 5 show the MSE performance of the TV-LMS algorithm and conventional LMS algorithm with different filter orders and different values of the parameter $C$. Again, both LMS algorithms provide similar performance results. The larger the parameter $C$ the less the MSE. It is clear that the TV-LMS algorithm performs much better than the conventional LMS algorithm in the low filter order region. Fig. 4, and Fig. 5 also show that both algorithms provide better MSE performance when the filter order increases.

Fig. 6 shows the MSE performance for the RLS ($\lambda = 1$) algorithm with different filter orders. As the RLS is highly sensitive to numerical instability [1, 2, 5], the filter order will severely affect the performance of the algorithm. Fig. 6 shows that there is no improvement in the performance of the RLS algorithm when the filter order increases. Its optimal filter order in this case is around 50. A careful selection of the filter order is therefore needed for optimal performance.

Fig 7 and Fig. 8 show the mean-squared error versus the number of samples $N$ for different algorithms. These figures provide information about the adaptive filter convergence time. It can be noticed that the RLS provides fastest convergence time; it also gives the best MSE performance when filter order is small. However, this comes at the cost of computation time needed for RLS as shown in Fig.9.
Figure 8: MSE vs. number of samples for different adaptive algorithms (SNR = 2 dB, $M = 100$, $f_o = 100$ Hz).

Figure 9: Computation time of different adaptive algorithms with different filter orders.

Fig. 9 shows the computational time for different algorithms with different filter orders. The computation time for the conventional LMS algorithm and the TV-LMS algorithm is relatively similar and much less than that of the RLS algorithm. The above figures also show that the RLS computation time increases rapidly and non-linearly with the filter order.

3.2 Quadratic frequency-modulated (QFM) signal

As non-linear FM signals are important in application, we assume here that the original signal $x(t)$ is a finite-length QFM signal of the form:

$$x(t) = \sin(\omega_0 t + \alpha t^2/2 + \beta t^3/3)\Pi_T(t - T)$$  \hspace{1cm} (15)

where $\omega_0 = 2\pi f_o$ is a constant (initial frequency), $T$ is the signal duration, and $\alpha$ and $\beta$ are the modulation indices which determine the bandwidth of the QFM signal.

The bandwidth $BW$ of this QFM signal can be adjusted by varying the parameters $\alpha$ and $\beta$. Increasing $\alpha$ and $\beta$ will result in increasing the signal bandwidth, as can be numerically shown using the relationships [3]:

$$f_m = \frac{1}{2\pi} \int_0^\infty \omega |X(\omega)|^2 d\omega$$  \hspace{1cm} (16)

$$BW = \frac{1}{2\pi} \int_0^\infty (\omega - \omega_m)^2 |X(\omega)|^2 d\omega$$  \hspace{1cm} (17)

where $X(f)$ is the Fourier transform of $x(t)$ and $f_m$ is its mean frequency. Fig. 10 shows the spectrum of a QFM narrowband signal with $f_o = 100$ Hz, $\alpha = 0.5$, and $\beta = 0.37$. In this case the bandwidth is 50 Hz.

Figure 10: Spectrum of a QFM narrowband signal with $f_o = 100$ Hz, $\alpha = 0.5$, and $\beta = 0.37$. In this case the bandwidth is 50 Hz.

Fig. 11 and Fig. 12 show the mean-squared error (MSE) performance of the conventional LMS algorithm and the time-varying LMS (TV-LMS) algorithm using different filter orders and different QFM bandwidths. Again, results show that the TV-LMS algorithm performs better than LMS algorithm, which is similar to result shown in the previous section for noise reduction in single-tones (Fig. 2). Comparing the bandwidths in Fig. 11 and Fig. 12, the performance for both algorithms decreases slightly when the QFM signal bandwidth increases. The results are still in an acceptable range, therefore, the algorithm can be used for narrowband signals with similar performance curves. Fig. 11 and Fig. 12 also indicate that increasing the filter order does not provide much improvement in the MSE.
Figure 11: MSE performance for the LMS and TV-LMS algorithms with different \( \mu_o \), filter order \( M = 30 \), and different bandwidths (SNR = 2 dB, \( f_o = 100 \text{ Hz} \)).

Figure 12: MSE performance for the LMS and TV-LMS algorithms with different \( \mu_o \), filter order \( M = 100 \), and different bandwidths (SNR = 2 dB, \( f_o = 100 \text{ Hz} \)).

Figure 13: MSE for the LMS algorithm with different filter orders in QFM signal BW=50 Hz (SNR = 2 dB, \( a = 0.01, b = 0.7 \)).

Figure 14: MSE for the LMS algorithm with different filter orders in QFM signal BW=50 Hz (SNR = 2 dB, \( C = 2, a = 0.01, b = 0.7 \)).

performance.

Fig. 13, and Fig. 14 show the MSE performance of the conventional LMS algorithm and the TV-LMS algorithm with different filter orders and different values of the \( C \) parameter. In general, results show that the performance will improve when the filter order increases. However, this is not true for \( \mu \) values that are too far from the optimal range. Fig. 13 and Fig. 14 also show that TV-LMS performs better with larger \( C \). Again, this is in accord with Fig. 4 and Fig. 5.

Fig. 15 shows the MSE performance for the RLS (\( \lambda = 1 \)) algorithm with different filter orders. Fig. 15 provides an identical conclusion for the RLS algorithm as in Fig. 6: RLS shows no improvement when the filter order increases. Compared with Fig. 13 and Fig. 14, Fig. 15 also shows that the RLS algorithm performs worse than conventional LMS algorithm and the time-varying LMS algorithm in this nonlinear QFM environment. It can be concluded that the TV-LMS algorithm provides a computation time close to that of the conventional LMS algorithm, with the best MSE performance in QFM signal environments.

Fig. 16 and Fig. 17 show the mean-squared error versus the number of samples \( N \). In Fig. 16, the RLS algorithm can provide a fast converging speed. However, it starts to fall apart and its MSE performance worsens as time passes. Hence, the RLS does not provide the same benefits for QFM signal
as it did for a single-tone sinusoid (Fig. 7 and Fig. 8). In general, it is shown that the TV-LMS algorithm has faster convergence than the LMS for both single-tone sinusoid (Fig. 7 and Fig. 8) and QFM signal (Fig. 16 and Fig. 17).

3 Conclusion

In this work we proposed a time-varying LMS (TV-LMS) algorithm and presented a comprehensive study of its performance as compared to other well-known algorithms: the conventional LMS algorithm and the recursive least-squared (RLS) algorithm. The study concentrated selected noise reduction in single-tone sinusoids and quadratic FM signals. Four performance criteria are used in this comparison: the algorithm execution time, the minimum mean-squared error (MSE), the filter order, and the algorithm convergence speed. The TV-LMS algorithm with a decaying time-varying law for the convergence parameter was shown to have better performance than the conventional LMS algorithm in terms of faster convergence and less mean squared error.

For noise reduction in a single-tone sinusoid with additive white Gaussian noise (AWGN), larger filter order will provide better MSE performance for both conventional LMS and TV-LMS algorithms. However, this does not apply to the RLS algorithm. Increasing the filter order in RLS algorithm does not result in any improvement in the MSE performance. Simulations also showed that the TV-LMS algorithm provides better MSE performance than the conventional LMS and the RLS algorithms with higher filter order (e.g. \( M=100 \)). However, the RLS algorithm provides the best MSE performance when small filter order is used (e.g. \( M=30 \)). The filter order also affects the computation time and the convergence speed of the algorithms. Increasing the filter order will rapidly increase the computation time for the RLS algorithm, whereas this does not apply to the conventional LMS and the time-varying algorithms. In terms of convergence speed, the RLS algorithm still provides a faster convergence time. The TV-LMS algorithm can also provide a similar convergence speed with higher filter order and larger value for parameter \( C \). For a single-tone signal, the TV-LMS algorithm with high filter order and larger \( C \) value will be the best algorithm compared to the RLS algorithm and the conventional LMS algorithm in terms of computation time, convergence speed.
and MSE performance.

The above algorithms are also tested for noise reduction in a narrow-band QFM signal with AWGN. Results show that the proposed TV-LMS provides better MSE performance and faster convergence than the conventional LMS algorithm. The RLS algorithm under this condition gives the worst MSE performance and shows no benefit when increasing the filter order.

References


