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The Electromagnetic Field Evolution in the Presence of an Ellipsoid Shell

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Abstract

Eddy current based sensors allow an accurate analysis of conductive materials structures. Electromagnetic field evolution in the presence of cylindrical, spherical and rectangular bodies is described using analytical methods [1], [2]. The present paper offers an insight into the interaction of a steady state electromagnetic field with an ellipsoid shell. The magnetic field evolution equations are formulated and their solutions are proposed using the method of separation of variables.

Keywords: electromagnetic field, steady state, equations, ellipsoid shell

1 Introduction

An ellipsoid shell of interior wall coordinate $\xi = \xi_0 = const.$, exterior wall coordinate $\xi = m\,\xi_0 = const.$, and focal distance $f(0,c,0)$ is considered. The wall has constant conductivity σ and permeability μ and is submitted to a steady state electromagnetic field (Fig.1). An ellipsoidal system of coordinates (flat ellipsoid), ξ, φ, ψ, with the versors e_ξ, e_φ, e_ψ, is used, and the associated LAMÉ coefficients are:

$$L_1 = c\sqrt{\cos^2\varphi + \mathrm{sh}^2\xi} = L_2$$
$$L_3 = c\ \mathrm{ch}\,\xi \sin\varphi \qquad (1)$$

Since outside the ellipsoid wall the conductivity as well as the displacement current are assumed null, the magnetic field equations may be written:

$$\mathrm{div}\ \overline{H} = 0,\ \ \mathrm{curl}\ \overline{H} = 0 \qquad (2)$$

The vector field $\overline{H}$ being irrotational, it implies the existence of a scalar potential, X, so that:

$$\overline{H} = -\mathrm{grad}\,X \qquad (3)$$

and hence:

$$\mathrm{div}\ \overline{H} = \mathrm{div}\,(\mathrm{grad}\,X) = \Delta X = 0 \qquad (4)$$

The laplacean of the scalar potential X is:

$$\Delta X = \frac{1}{L_1 L_2 L_3}\left[\frac{\partial}{\partial\xi}\left(\frac{L_2 L_3}{L_1}\frac{\partial X}{\partial\xi}\right) + \frac{\partial}{\partial\varphi}\left(\frac{L_1 L_3}{L_2}\frac{\partial X}{\partial\varphi}\right) + \frac{\partial}{\partial\psi}\left(\frac{L_1 L_2}{L_3}\frac{\partial X}{\partial\psi}\right)\right] =$$
$$= \frac{1}{c^3(\mathrm{sh}^2\xi + \cos^2\varphi)\,\mathrm{ch}\,\xi\sin\varphi}\left[\frac{\partial}{\partial\xi}\left(c\,\mathrm{ch}\,\xi\sin\varphi\frac{\partial X}{\partial\xi}\right) + \right. \qquad (5)$$
$$\left. + \frac{\partial}{\partial\varphi}\left(c\,\mathrm{ch}\,\xi\sin\varphi\frac{\partial X}{\partial\varphi}\right) + \frac{\partial}{\partial\psi}\left(c\,\frac{\mathrm{sh}^2\xi + \cos^2\varphi}{\mathrm{ch}\,\xi\sin\varphi}\frac{\partial X}{\partial\psi}\right)\right]$$

2 The External Magnetic Field Intensity

Due to the fact that the magnetic field intensity, $\overline{H}$, is independent with respect to the ψ coordinate, being parallel to the Oz axis and uniformly located around the ellipsoid, namely:

$$\overline{H} = H_\xi\ e_\xi + H_\varphi\ e_\varphi \qquad (6)$$

taking into account relation (3) it results:

$$\frac{\partial X}{\partial\psi} = 0 \qquad (7)$$

Considering also (5) and (7), equation (4) becomes:

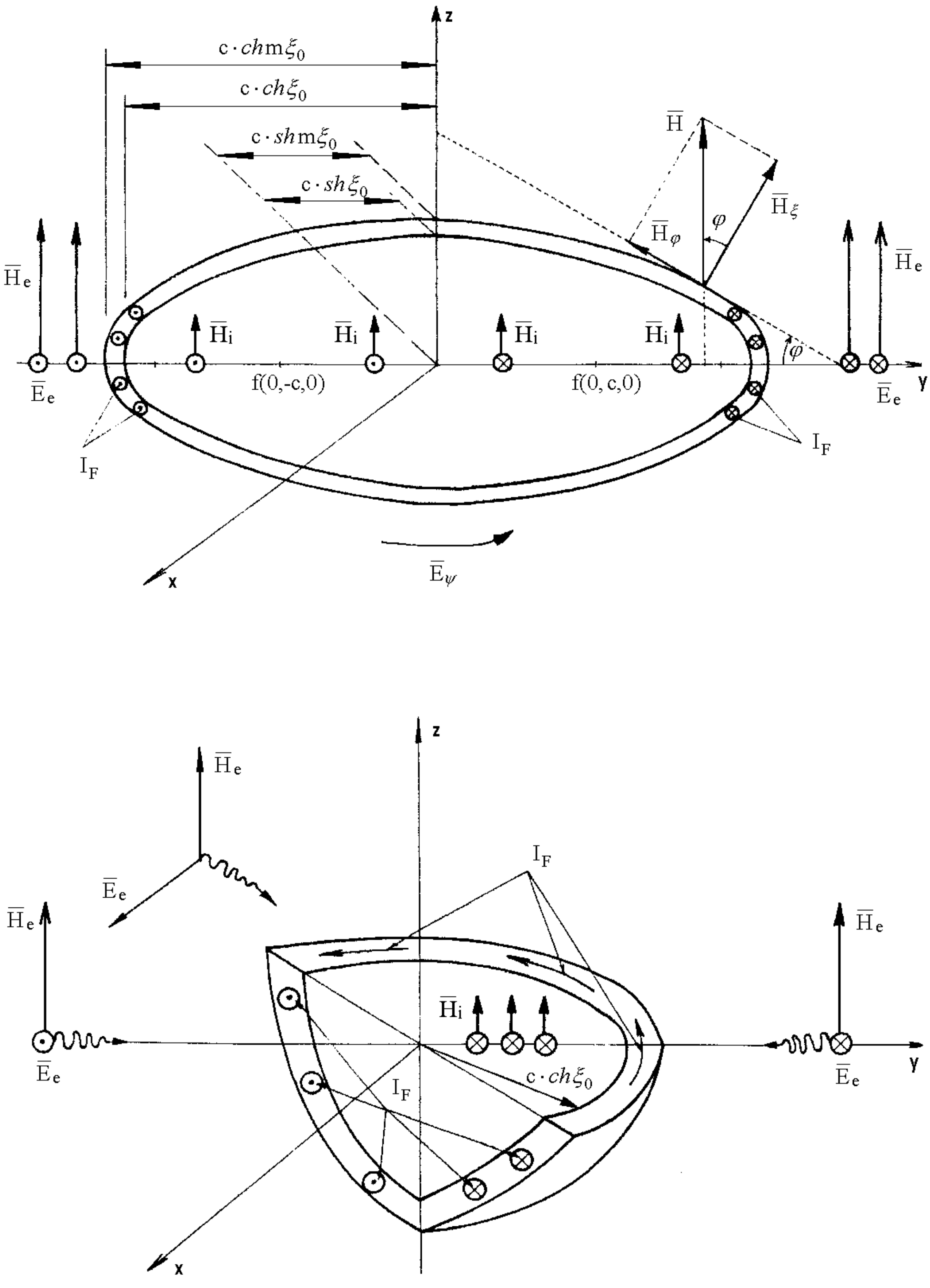

Figure 1: Ellipsoid shell cross sections

$$\frac{1}{c^2(\text{sh}^2\xi+\cos^2\varphi)\text{ch}\,\xi}\left[\frac{\partial}{\partial\xi}\left(\text{ch}\,\xi\frac{\partial X}{\partial\xi}\right)\right]+\frac{1}{c^2[\text{sh}^2\xi+\cos^2\varphi)\sin\varphi}\left[\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial X}{\partial\varphi}\right)\right]=0 \quad (8)$$

Because the term $1/[c^2(\text{sh}^2\xi+\cos^2\varphi)]$ is always positive (8) may be written:

$$\frac{1}{\text{ch}\,\xi}\left[\frac{\partial}{\partial\xi}\left(\text{ch}\,\xi\frac{\partial X}{\partial\xi}\right)\right]+\frac{1}{\sin\varphi}\left[\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial X}{\partial\varphi}\right)\right]=0 \quad (9)$$

Equation (9) is solved using the method of separation of variables and, consequently, a solution of the kind:

$$X(\xi,\varphi)=R(\xi)\,S(\varphi) \quad (10)$$

is investigated.

Replacing the proposed solution in (9), it follows:

$$\frac{S(\varphi)}{\text{ch}\,\xi}\left(\text{sh}\,\xi\frac{\partial R(\xi)}{\partial\xi}+\text{ch}\,\xi\frac{\partial^2 R(\xi)}{\partial\xi^2}\right)+\frac{R(\xi)}{\sin\varphi}\left(\cos\varphi\frac{\partial S(\varphi)}{\partial\varphi}+\sin\varphi\frac{\partial^2 S(\varphi)}{\partial\varphi^2}\right)=0 \quad (11)$$

Dividing by $S(\varphi)\,R(\xi)$ (11) becomes:

$$\frac{1}{R(\xi)\,\text{ch}\,\xi}\left(\text{sh}\,\xi\frac{\partial R(\xi)}{\partial\xi}+\text{ch}\,\xi\frac{\partial^2 R(\xi)}{\partial\xi^2}\right)+ \quad (12)$$

$$+\frac{1}{S(\varphi)\sin\varphi}\left(\cos\varphi\frac{\partial S(\varphi)}{\partial\varphi}+\sin\varphi\frac{\partial^2 S(\varphi)}{\partial\varphi^2}\right)=0$$

The unique possibility for the sum of two functions of different variables to be zero unrestrictedly implies they must be equal to a constant, namely:

$$\frac{1}{R(\xi)\,\text{ch}\,\xi}\left(\text{sh}\,\xi\frac{\partial R(\xi)}{\partial\xi}+\text{ch}\,\xi\frac{\partial^2 R(\xi)}{\partial\xi^2}\right)=\lambda \quad (13)$$

$$\frac{1}{S(\varphi)\sin\varphi}\left(\cos\varphi\frac{\partial S(\varphi)}{\partial\varphi}+\sin\varphi\frac{\partial^2 S(\varphi)}{\partial\varphi^2}\right)=-\lambda \quad (14)$$

Thus, there have been obtained two ordinary differential equations of second order, whose solutions are [3]:

$$R(\xi)=C_1\,\text{sh}\,\xi+C_2\,\text{sh}\,\xi\ \text{artg}(\text{sh}\,\xi)+C_2 \quad (15)$$

and

$$S(\varphi)=C_3\ \cos\varphi \quad (16)$$

The expression of the gradient in ellipsoidal coordinates for the scalar potential X, taking into account both the expressions found for the LAMÉ coefficients (1) and the condition (7) is:

$$\text{grad}\ X=\frac{1}{L_1}\frac{\partial X}{\partial\xi}\,e_\xi+\frac{1}{L_2}\frac{\partial X}{\partial\varphi}\,e_\varphi+\frac{1}{L_3}\frac{\partial X}{\partial\psi}\,e_\psi=$$

$$=\frac{1}{c\sqrt{\cos^2\varphi+\text{sh}^2\xi}}\frac{\partial X}{\partial\xi}\,e_\xi+\frac{1}{c\sqrt{\cos^2\varphi+\text{sh}^2\xi}}\frac{\partial X}{\partial\varphi}\,e_\varphi \quad (17)$$

Considering (15) and (16), the solution of equation (9) becomes:

$$X(\xi,\varphi)=R(\xi)\ S(\varphi)=[C_1\ \text{sh}\,\xi+C_2\ \text{sh}\,\xi\ \text{artg}(\text{sh}\,\xi)+C_2]\ C_3\cos\varphi \quad (18)$$

or

$$X(\xi,\varphi)=R(\xi)\ S(\varphi)=[C_{1'}\ \text{sh}\,\xi+C_{2'}\ \text{sh}\,\xi\ \text{artg}(\text{sh}\,\xi)+C_{2'}]\ \cos\varphi \quad (19)$$

where $C_{1'}=C_1C_3$, $C_{2'}=C_2C_3$.

From (3) and (17) it follows:

$$\overline{H}=-\,\text{grad}\ X=-\frac{1}{c\sqrt{\cos^2\varphi+\text{sh}^2\xi}}\frac{\partial X}{\partial\xi}\,e_\xi-\frac{1}{c\sqrt{\cos^2\varphi+\text{sh}^2\xi}}\frac{\partial X}{\partial\varphi}\,e_\varphi \quad (20)$$

Identifying (20) with (6) and considering that

$$\frac{\partial X}{\partial\xi}=[\,C_{1'}\,\text{ch}\,\xi+C_{2'}\,\text{ch}\,\xi\ \text{artg}(\text{sh}\,\xi)+C_{2'}\,\text{sh}\,\xi\ \ \text{th}\,\xi]\ \cos\varphi$$

$$\frac{\partial X}{\partial\varphi}=-[C_{1'}\,\text{sh}\,\xi+C_{2'}\,\text{sh}\,\xi\ \text{artg}(\text{sh}\,\xi)+C_{2'}]\ \sin\varphi\ ,$$

it is obtained:

$$H_\xi(\xi,\varphi)=-\frac{1}{c\sqrt{\cos^2\varphi+\text{sh}^2\xi}}\frac{\partial X}{\partial\xi}=-\frac{[C_{1'}\,\text{ch}\,\xi+C_{2'}\,\text{ch}\,\xi\ \text{artg}(\text{sh}\,\xi)+C_{2'}\ \text{th}\,\xi]\ \cos\varphi}{c\sqrt{\cos^2\varphi+\text{sh}^2\xi}}$$
$$H_\varphi(\xi,\varphi)=-\frac{1}{c\sqrt{\cos^2\varphi+\text{sh}^2\xi}}\frac{\partial X}{\partial\varphi}=\frac{[C_{1'}\,\text{sh}\,\xi+C_{2'}\,\text{sh}\,\xi\ \text{artg}(\text{sh}\,\xi)+C_{2'}\ ]\ \sin\varphi}{c\sqrt{\cos^2\varphi+\text{sh}^2\xi}} \quad (21)$$

Due to the existence of eddy currents I_F in the ellipsoid shell, it appears a so called "reaction field", characterized by the " reaction factor", $\overline{W}$, analogous to a reflexion coefficient. In the exterior of the volume where a "reaction field" is present, the magnetic field, $\overline{H}$, is constant, unmodified, and $\overline{W}=0$, whereas in the interior of the volume affected by the "reaction field", the magnetic field, $\overline{H}$, is modified, and $\overline{W}\neq 0$.

Taking into account that far from the shield, in the zone where $\overline{W}=0$, the relation:

$$H=\sqrt{H_\xi^2+H_\varphi^2}=H_e \quad (22)$$

is true, the following expressions are proposed for the constants $C_{1'}$ and $C_{2'}$:

$$C_{1'} = c\,H_e \quad C_{2'} = c\,W\,H_e \tag{23}$$

Therefore, for the outside field it results the final solution:

$$H_\xi(\xi,\varphi) = -\frac{[\mathrm{ch}\,\xi + W\,\mathrm{ch}\,\xi\,\mathrm{artg}(\mathrm{sh}\,\xi) + W\,\mathrm{th}\,\xi]\cos\varphi}{\sqrt{\cos^2\varphi + \mathrm{sh}^2\xi}}\,H_e$$

$$H_\varphi(\xi,\varphi) = \frac{[\mathrm{sh}\,\xi + W\,\mathrm{sh}\,\xi\,\mathrm{artg}(\mathrm{sh}\,\xi) + W]\sin\varphi}{\sqrt{\cos^2\varphi + \mathrm{sh}^2\xi}}\,H_e \tag{24}$$

from where the "reaction field" may be inferred:

$$H_{\xi(\mathrm{reaction})} = \frac{[\mathrm{ch}\,\xi\,\mathrm{artg}(\mathrm{sh}\,\xi) + \mathrm{th}\,\xi]\cos\varphi}{\sqrt{\cos^2\varphi + \mathrm{sh}^2\xi}}\,W\,H_e$$

$$H_{\varphi(\mathrm{reaction})} = -\frac{[\mathrm{sh}\,\xi\,\mathrm{artg}(\mathrm{sh}\,\xi) + 1]\sin\varphi}{\sqrt{\cos^2\varphi + \mathrm{sh}^2\xi}}\,W\,H_e \tag{25}$$

Obviously, the components of the magnetic field intensity far from the shield, in the unperturbed zone with $\overline{W} = 0$, are:

$$H_\xi(\xi,\varphi) = -\frac{\mathrm{ch}\,\xi\cos\varphi}{\sqrt{\cos^2\varphi + \mathrm{sh}^2\xi}}\,H_e$$

$$H_\varphi(\xi,\varphi) = \frac{\mathrm{sh}\,\xi\sin\varphi}{\sqrt{\cos^2\varphi + \mathrm{sh}^2\xi}}\,H_e \tag{26}$$

3 The Magnetic Field Intensity Inside the Shell

The magnetic field intensity calculus for the interior of the ellipsoid shell is analogous with the one made for the exterior. Due to the fact that in the unperturbed interior zone, with $\overline{W} = 0$, the relation $H = \sqrt{H_\xi^2 + H_\varphi^2} = H_i = FH_e$ must be fulfilled (F – screening factor), taking into account both (21) and the fact that $\overline{W} = 0$, proposing for the constants $C_{1'}$ and $C_{2'}$ the relations $C_{1'} = c\,H_i$, $C_{2'} = c\,W\,H_i$, the inner magnetic field intensity $\overline{H}_i$ has the components:

$$H_\xi(\xi,\varphi) = -\frac{\mathrm{ch}\,\xi\cos\varphi}{\sqrt{\cos^2\varphi + \mathrm{sh}^2\xi}}\,F\,H_e$$

$$H_\varphi(\xi,\varphi) = \frac{\mathrm{sh}\,\xi\sin\varphi}{\sqrt{\cos^2\varphi + \mathrm{sh}^2\xi}}\,F\,H_e \tag{27}$$

4 Conclusions

A study of an ellipsoid shell submitted to the action of a steady state electromagnetic field is performed. The expressions for the magnetic field intensity in the exterior (24) and in the interior (27) of the shell are found. The homogeneous equations describing the magnetic field evolution are solved using the method of separation of variables. The theoretical results may be used in devising methods and associated systems and procedures in underground objects detections and magnetic field shielding.

5 References

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